On Some Elliptic Problems Involving Powers of the Laplace Operator

by

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"Sometimes science is more art than science, Morty.
A lot of people don't get that."

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Published and Submitted Content

Chapter 3 is composed entirely by the results contained in the paper

- E. Colorado, A. Ortega, *The Brezis-Nirenberg problem for the fractional Laplacian with mixed Dirichlet-Neumann boundary conditions.* Preprint arXiv:1805.10093v2,

which I co-authored jointly with one of my advisors, Professor E. Colorado. This work was submitted to the scientific repository Arxiv.org. The material from this source included in this thesis is not singled out with typographic means and references.

Contents

Nomenclature

Symbol	**Meaning**
$x=(x_1,x_2,...,x_N)$	Element of $\mathbb{R}^n$
$r=\vert x\vert=\sqrt{(x_1^2+x_2^2+\cdots+x_N^2)}$	Modulus of x
$\langle\cdot,\cdot\rangle$	Scalar product in $\mathbb{R}^N$
$\partial_i u=\dfrac{\partial u}{\partial x_i}=u_{x_i}$	Partial derivative of u respecto to x_i
$\partial_{ij}^2 u=\dfrac{\partial^2 u}{\partial x_i\partial x_j}=u_{x_ix_j}$	Second partial derivative of u respect to x_i and x_j
$\nabla u=\left(\dfrac{\partial u}{\partial x_1},\dfrac{\partial u}{\partial x_2},\ldots,\dfrac{\partial u}{\partial x_N}\right)$	Gradient of u
$\frac{\partial u}{\partial \nu}=\langle\nabla u,\nu\rangle$	Outwards normal (to $\partial\Omega$) derivative
$\Delta u=\operatorname{div}(\nabla u)$	Laplacian of u
$(-\Delta)^s u$	Spectral Fractional Laplacian of u
$\frac{\partial w}{\partial \nu^s}$	$-\kappa_s\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}$
κ_s	Normalizing constant equals to $\frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$
$2_s^*=\dfrac{2N}{N-2s}$	Critical fractional Sobolev exponent
$\partial\Omega$	Boundary of Ω
$\mathscr{C}_\Omega=\Omega\times(0,\infty)$	Extension cylinder of Ω
$\partial_L\mathscr{C}_\Omega=\partial\Omega\times[0,\infty)$	Lateral boundary of $\mathscr{C}_\Omega$
$B_R(x_0)$	Ball in $\mathbb{R}^N$ centered at x_0 with radius R
$\vert A\vert$	Lebesgue measure of $A\subset\mathbb{R}^N$
$\vert A\vert_\omega$	Measure of $A\subset\mathbb{R}^N$ respect to the measure $d\mu=\omega dx$
χ_A	Characteristic function of the set A
$\Vert\cdot\Vert_X$	Norm in the space X
X'	Dual space of X
$\setminus$	Difference of sets
δ_{x_0}	Dirac's delta centered at x_0
δ_{ij}	Kronecker's delta
$a.e.$	Almost everewhere
$u^+=\max\{u,0\}$	Positive part of the function u
$u^-=\max\{-u,0\}$	Negative part of the function u
$\mathcal{C}(\Omega)$ or $\mathcal{C}^0(\Omega)$	Continuous functions in Ω
$\mathcal{C}_0(\Omega)$	Continuous functions in Ω with compact support
$\mathcal{C}^{0,\gamma}(\Omega)=\mathcal{C}^\gamma(\Omega)$	Hölder continuous functions in Ω with exponent γ
$\vert u\vert_\gamma=\sup\limits_{\substack{x,y\in\Omega\\ x\neq y}}\dfrac{\vert u(x)-u(y)\vert}{\vert x-y\vert^\gamma}$	Seminorm in the space $\mathcal{C}^\gamma(\Omega)$
$\Vert u\Vert_{\mathcal{C}^\gamma(\Omega)}=\Vert u\Vert_{\mathcal{C}(\Omega)}+\vert u\vert_\gamma$	Norm in the space $\mathcal{C}^\gamma(\Omega)$

Symbol	**Meaning**
$\mathcal{C}^k(\Omega)$	Functions of class k in Ω
$\mathcal{C}_0^k(\Omega)$	Functions of class k in Ω with compact support
$\mathcal{C}^\infty(\Omega)$	Functions infinitely differentiable in Ω
$\mathcal{C}_0^\infty(\Omega) = \mathcal{D}(\Omega)$	Functions in $\mathcal{C}^\infty(\Omega)$ with compact support
$\mathcal{D}'$	Dual space of $\mathcal{C}_0^\infty(\Omega)$, i.e. the space of distributions
$L^p(\Omega),\ 1 \le p < \infty$	$\left\{u : \Omega \mapsto \mathbb{R} : u \text{ measurable},\ \int_\Omega \|u\|^p dx\right\}$
$L^\infty(\Omega)$	$\{u : \Omega \mapsto \mathbb{R} : u \text{ measurable and } \|u(x)\| \le C \ a.e. \text{ in } \Omega\}$
$H^1(\Omega)$	Completeness of $\mathcal{C}_0^\infty(\Omega)$ with the norm $\\|\phi\\|_{H^1(\Omega)} = \\|\phi\\|_{L^2(\Omega)} + \\|\nabla\phi\\|_{L^2(\Omega)}$
$H^s(\Omega)$	$\left\{u = \sum_j a_j\varphi_j \in L^2(\Omega) : \\|u\\|_{H^s(\Omega)} = \sum_j \lambda_j^s a_j^2 < \infty\right\}$
$H^s_{\Sigma_\mathcal{D}}(\Omega)$	$\{u \in H^s(\Omega) : u = 0 \text{ on } \Sigma_\mathcal{D} \subset \partial\Omega\}$
$\mathcal{X}_0^s(\mathscr{C}_\Omega)$	Completeness of $\mathcal{C}^\infty_{0,L}(\mathscr{C}_\Omega)$ with the norm $\\|\phi\\|_{\mathcal{X}_0^s(\mathscr{C}_\Omega)} = \left(\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\|\nabla\phi\|^2 dxdy\right)^{1/2}$
$\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)$	Completeness of $\mathcal{C}_0^\infty\big((\Omega \cup \Sigma_\mathcal{N}) \times [0,\infty)\big)$ with the norm $\\|\phi\\|_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)} = \left(\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\|\nabla\phi\|^2 dxdy\right)^{1/2}$
$S(s,N)$	Sobolev constant equals to $\frac{2\pi^s\Gamma(1-s)\Gamma(\frac{N+2s}{2})(\Gamma(\frac{N}{2}))^{\frac{2s}{N}}}{\Gamma(s)\Gamma(\frac{N-2s}{2})(\Gamma(N))^s}$
$\widetilde{S}(\Sigma_\mathcal{D})$	Sobolev constant relative to $\Sigma_\mathcal{D} \subset \partial\Omega$ equals to $\inf\limits_{\substack{u \in H^s_{\Sigma_\mathcal{D}}(\Omega) \\ u \not\equiv 0}} \frac{\\|u\\|^2_{H^s_{\Sigma_\mathcal{D}}(\Omega)}}{\\|u\\|^2_{L^{2^*_s}(\Omega)}}$
$\widetilde{S}(\Sigma_\mathcal{N})$	Sobolev constant relative to $\Sigma_\mathcal{N} \subset \partial\Omega$ equals to $2^{-\frac{2s}{N}} S(s,N)$

Summary of contents

This PhD Thesis is devoted to the study of some elliptic problems involving powers of the positive Laplace operator. In general, these problems could be written as

$$(P) \qquad \left\{ \begin{array}{ll} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ + \text{Boundary conditions} & \text{on } \partial\Omega, \end{array} \right.$$

for a smooth bounded domain $\Omega \subset \mathbb{R}^N$. The operator $(-\Delta)^s$, referred to as the *spectral fractional Laplacian*, is one of the so called fractional Laplace operators. As its very name suggests, the spectral fractional Laplacian is the one defined via the spectral decomposition of the Laplace operator under the particular boundary condition imposed. Indeed, if (φ_i, λ_i) are the eigenfunctions (normalized with respect to the $L^2(\Omega)$-norm) and the eigenvalues of $(-\Delta)$, the action of the positive Laplace operator on a function,

$$u(x) = \sum_i \langle u, \varphi_i \rangle \varphi_i(x), \quad x \in \Omega,$$

is given by the action on each eigenfunction,

$$(-\Delta)u(x) = \sum_i \langle u, \varphi_i \rangle (-\Delta)\varphi_i(x) = \sum_i \lambda_i \langle u, \varphi_i \rangle \varphi_i(x), \quad x \in \Omega.$$

Then, it is natural to define the operator $(-\Delta)^s$, $0 < s < 1$ as the operator whose action is given by

$$(-\Delta)^s u(x) = \sum_i \lambda_i^s \langle u, \varphi_i \rangle \varphi_i(x), \quad x \in \Omega.$$

The common thread among the problems studied in this work is, at one hand, their *critical nature*, in the sense that we will deal with:

I. Problems with a lack of regularity.
II. Problems with a lack of compactness.

On the other hand, the problems studied in this work can be also classified according to the imposed boundary conditions, namely,

1. Problems with Mixed Dirichlet-Neumann Boundary data.
2. Problems with Dirichlet Boundary data.

By Mixed Dirichlet-Neumann Boundary data we mean

$$B(u) = \chi_{\Sigma_{\mathcal{D}}} \cdot u + \chi_{\Sigma_{\mathcal{N}}} \cdot \frac{\partial u}{\partial \nu},$$

where ν is the outwards normal vector to $\partial\Omega$, $\chi_A(x)$ stands for the characteristic function of the set A and $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$. In particular, we assume that $\Sigma_{\mathcal{D}}$ is a closed manifold of positive $(N-1)$-dimensional Lebesgue measure, $|\Sigma_{\mathcal{D}}| = \alpha \in (0, |\partial\Omega|)$, and

$$\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset\,, \quad \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega \quad \text{and} \quad \Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma,$$

with Γ a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$. Much is known about Dirichlet and Neumann boundary problems associated with elliptic equations as

$$-\Delta u = f(x,u) \tag{0.1}$$

with different nonlinearities $f(x,u)$. In contrast, mixed Dirichlet-Neumann boundary problems have been much less investigated. Nevertheless, some important results dealing with mixed Dirichlet-Neumann boundary problems associated with (0.1) have been proved over the years. See [**1, 2, 34, 35, 39, 40, 41, 61, 68, 67, 80**] among others.
Problems associated with (0.1), substituting the operator by the fractional Laplacian, have been extensively investigated in the last years, with Dirichlet or Neumann boundary conditions (cf., e.g. [**18, 19, 24, 29, 27, 42, 44, 69, 79, 84**] among others). However, these fractional elliptic problems, once again, have not been so much investigated with mixed Dirichlet-Neumann boundary data; cf. [**20, 36**]. Indeed, up to our knowledge, there are no references for mixed Dirichlet-Neumann boundary problems involving the spectral fractional Laplacian operator, which is the one we deal with and focus on this PhD Thesis.
On the other hand, although mixed boundary value problems and Dirichlet boundary value problems share some important qualitative properties, the study of mixed problems presents unique particularities that make them of considerable interest. Examples of these particularities are:

- Solutions of mixed boundary data problems are less regular than solutions of the same problems under Dirichlet boundary conditions. Indeed, there is an upper limit for the regularity in terms of Hölder continuity; (see Chapter 1).
- Moving the boundary condition so that the Dirichlet part of the boundary becomes small enough, the existence of solutions to a certain critical problem can be proved in contrast to non-existence results for the same critical problems under Dirichlet boundary condition; (see Chapter 3).

The lack of regularity mentioned above is a typical phenomena when dealing with elliptic problems endowed with mixed boundary conditions; see for instance [**77**], where it is obtained the optimal regularity for nontrivial solutions of mixed Dirichlet-Neumann elliptic problems which, indeed, is $\mathcal{C}^{\frac{1}{2}}$ up to the boundary.
In addition to the boundary condition, the specific problems studied here are determined by the right-hand side $f(x,u)$ in (P). In particular, in this work we will consider as a right-hand side term the following,

a) A summable function $f(x) \in L^p(\Omega)$, with $p > \frac{N}{2s}$.
b) A concave-convex function involving subcritical powers, $f(u) = \lambda u^q + u^r$ with $\lambda > 0$ and $0 < q \leq 1 < r < 2_s^*$.
c) A critical power function and a linear term, $f(u) = \lambda u + u^{2_s^*-1}$ with $\lambda \geq 0$.
d) A power function involving up to critical powers together with an inverse operator, $f(u) = \lambda(-\Delta)^{-\beta}u + |u|^{r-1}u$ with $\lambda > 0$, $1 < r \leq 2_s^* - 1$ and an appropriate $\beta > 0$.

The main purpose of this work is then outlined as follows. First, we study the fractional mixed boundary value problems for the subcritical range of exponents. To this end we focus on:

1.1. Study the regularity properties of solutions of problem (P) with mixed Dirichlet-Neumann boundary condition and a right-hand side given as in item a). Study the behavior of such solutions when we move the boundary condition in a way to be specified later. Prove uniform estimates on the Hölder norm of solutions of such mixed linear fractional problems even when we move the boundary condition.

1.2. Study the existence and some qualitative properties of positive solutions of the mixed concave-convex problem obtained by considering problem (P) with mixed Dirichlet-Neumann boundary condition and a nonlinear concave-convex right-hand side given as above in item b). Characterize the existence of such positive solutions in terms of the size of the parameter $\lambda > 0$. Study the multiplicity of positive solutions and the behavior of some class of solutions when we move the boundary condition.

Next, we turn our attention to the study of the mixed critical problem obtained by considering problem (P) with mixed Dirichlet-Neumann boundary condition and a right-hand side term given as in item c). Our main objective at this point is,

2. Characterize the existence of positive solutions for these mixed critical problems in terms of the parameter $\lambda \geq 0$. Study the behavior of the positive solutions of the pure critical power problem obtained by taking $\lambda = 0$ when we move the boundary condition.

Once we have completed these steps, we continue with the study of problem (P) with Dirichlet boundary condition and a right-hand side given as in item d) for an appropriate $\beta > 0$. To accomplish this step, we now focus on:

3.1. Study the corresponding local problem obtained by setting $s = 1$ and $\beta = 1$. Characterize the existence of solutions in terms of the parameter $\lambda > 0$ for both, the subcritical and the critical problems.

3.2. Generalize the former results to the fractional framework and prove the existence of solutions to problem (P) with a right-hand side as in d) for both, the subcritical and the critical exponent cases, in terms of the parameter $\lambda > 0$.

As we will see, problems like (P) with a right-hand side given as in item d) arise when one studies the steady-states of certain high-order parabolic equations. To get closer to future extensions and analysis of similar high-order problems we conclude this work performing an homotopic study of a nonlinear high-order parabolic problem in divergence form.

This PhD Thesis dissertation is then divided into the following main parts:

Part 1. Subcritical Problems with Mixed Dirichlet-Neumann Boundary data.
Part 2. Critical Problems with Mixed Dirichlet-Neumann Boundary data.
Part 3. Critical Problems involving inverse operators and Dirichlet Boundary data.

Description of the results

The main aim of **Part 1** is to establish some existence results for a subcritical fractional elliptic concave-convex problem with mixed Dirichlet-Neumann boundary condition. To this end, we start **Chapter 1** studying some regularity properties of solutions for the following mixed boundary value problem,

$$(P^s) \qquad \left\{\begin{array}{ll} (-\Delta)^s u = f & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{array}\right.$$

where $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$, $p > \frac{N}{2s}$, Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$ and by $B(u)$ we denote the mixed boundary condition,

$$B(u) = \chi_{\Sigma_{\mathcal{D}}} \cdot u + \chi_{\Sigma_{\mathcal{N}}} \cdot \frac{\partial u}{\partial \nu},$$

where ν is the outwards normal vector to $\partial\Omega$, $\chi_A(x)$ stands for the characteristic function of the set A and $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ satisfy the hypotheses:

$$(\mathfrak{B}) \quad \left\{\begin{array}{l} \Sigma_{\mathcal{D}} \text{ and } \Sigma_{\mathcal{N}} \text{ are smooth } (N-1)\text{-dimensional submanifolds of } \partial\Omega. \\ \Sigma_{\mathcal{D}} \text{ is a closed manifold of positive } (N-1)\text{-dimensional Lebesgue measure,} \\ \qquad\qquad |\Sigma_{\mathcal{D}}| = \alpha \in (0, |\partial\Omega|). \\ \Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset\,,\ \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega \text{ and } \Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma \text{ where } \Gamma \text{ is a smooth} \\ (N-2)\text{-dimensional submanifold of } \partial\Omega. \end{array}\right.$$

The main result proved in this chapter is the following.

Theorem 1. *Assume that Ω is a smooth bounded domain of $\mathbb{R}^N$ such that $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ satisfy the hypotheses $(\mathfrak{B})$ and let u be the solution of problem (P^s) with $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. Then $u \in \mathcal{C}^\gamma(\overline{\Omega})$ for some $0 < \gamma < \frac{1}{2}$. Even more, there exists a constant $\mathscr{H} = \mathscr{H}(N, s, f, p, |\Sigma_{\mathcal{D}}|) > 0$ such that*

$$|u(x) - u(y)| \leq \mathscr{H} |x - y|^\gamma, \ \forall\ x, y \in \overline{\Omega}.$$

To prove Theorem 1 we follow some of the ideas in [**62, 80**]. Using the De Giorgi truncation method, Stampacchia (see [**80**]) established the regularity of solutions of the mixed boundary value problem involving the classical Laplace operator. Due to the nonlocal nature of problem (P^s), some difficulties arise when trying to apply this truncation method to solutions to (P^s). Based on the ideas of [**24, 27, 29**], at this point we will make full use of the local realization of the fractional operator $(-\Delta)^s$ in terms of certain auxiliary degenerate elliptic problem. We use the results of [**48**] to adapt the procedures of [**80**] to the case of degenerate elliptic equations with weights in the Muckenhoupt class A_2 (see [**48**] for the precise definition as well as some useful properties of those weights).
In addition to Theorem 1, following some ideas in [**34**], we conclude this chapter studying the behavior of problem (P^s) when we move the boundary condition in a regular way as specified next.
Given $I_\varepsilon = [\varepsilon, |\partial\Omega|]$ for some $\varepsilon > 0$, let us consider the family of closed sets $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha \in I_\varepsilon}$, satisfying the hypotheses:

(B_1) $\Sigma_{\mathcal{D}}(\alpha)$ has a finite number of connected components.
(B_2) $\Sigma_{\mathcal{D}}(\alpha_1) \subset \Sigma_{\mathcal{D}}(\alpha_2)$ if $\alpha_1 < \alpha_2$.
(B_3) $|\Sigma_{\mathcal{D}}(\alpha_1)| = \alpha_1 \in I_\varepsilon$.

We denote by $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega \backslash \Sigma_{\mathcal{D}}(\alpha)$, and we assume that $\Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha) = \Gamma(\alpha)$ is a $(N-2)$-dimensional smooth submanifold of $\partial\Omega$. For a family of this type we consider the corresponding family of mixed boundary value problems,

$$(P_\alpha^s) \qquad \left\{ \begin{array}{ll} (-\Delta)^s u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{array} \right.$$

where $B_\alpha(u)$ is the boundary condition associated to the parameter α in the previous hypotheses and the boundary manifolds $\Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}}(\alpha)$ satisfy the corresponding hypotheses $(\mathfrak{B}_\alpha)$. In this scenario we prove the following result.

Corollary 1. *Given Ω be a smooth bounded domain such that the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha \in I_\varepsilon}$ satisfies the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3), let u_α be the solution of (P_α^s) with $\frac{1}{2} < s < 1$ and $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. Then, there exists two constants $\mathscr{H} > 0$ and $0 < \gamma < \frac{1}{2}$ both independent of $\alpha \in I_\varepsilon$ such that*

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\Omega)} \leq \mathscr{H}.$$

We continue in **Chapter 2** with the study of the semilinear subcritical concave-convex problem,

$$(P_\lambda^s) \qquad \left\{ \begin{array}{ll} (-\Delta)^s u = \lambda u^q + u^r & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{array} \right.$$

with $\frac{1}{2} < s < 1$, $\lambda > 0$, $0 < q \leq 1 < r < \frac{N+2s}{N-2s}$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and

$$B(u) = \chi_{\Sigma_{\mathcal{D}}} \cdot u + \chi_{\Sigma_{\mathcal{N}}} \cdot \frac{\partial u}{\partial \nu},$$

with $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ satisfying hypotheses $(\mathfrak{B})$ above. Problems like (P_λ^s) have been studied in the last decades with the classical Laplace operator and Dirichlet boundary condition, c.f. [**9, 65**] or [**10**] for a deep study; with the Laplace operator and mixed Dirichlet-Neumann boundary conditions, c.f. [**1, 2, 34**]; with the p-Laplace operator, c.f. [**23, 54, 55**]; with fully nonlinear operators, c.f. [**32**]; and more recently with the fractional Laplace operator and Dirichlet boundary condition, c.f. [**18, 19, 24**]. Up to our knowledge, this is the first work where the concave-convex problem is analyzed with the spectral fractional Laplacian associated with mixed Dirichlet-Neumann boundary conditions.
The main result to be proven in this chapter is stated as follows.

Theorem 2. *Assume that $\frac{1}{2} < s < 1$, $N > 2s$ and $0 < q \leq 1 < r < \frac{N+2s}{N-2s}$. Then*

(1) If $q = 1$ there exists at least one solution of (P_λ^s) for every $0 < \lambda < \lambda_{1,s}$, where $\lambda_{1,s}$ denotes the first eigenvalue of the spectral fractional Laplacian with homogeneous mixed Dirichlet-Neumann boundary condition. There is no solution for $\lambda \geq \lambda_{1,s}$. Even more, there is a branch of solutions to (P_λ) bifurcating from $(\lambda_{1,s}, 0)$, which cuts the axis $\{\lambda = 0\}$.

(2) If $0 < q < 1$ there exists $0 < \Lambda < \infty$ such that:
- *(a) If $0 < \lambda < \Lambda$ there is a minimal solution of (P_λ^s). Moreover, the family of minimal solutions is increasing with respect to λ.*
- *(b) If $\lambda = \Lambda$ there is at least one solution of (P_λ^s).*
- *(c) If $\lambda > \Lambda$ there is no solution of (P_λ^s).*
- *(d) Problem (P_λ^s) admits at least two solutions for every $0 < \lambda < \Lambda$.*

The next result deals with the sublinear case $0 < q < 1$ and provides us with uniform $L^\infty(\Omega)$-bounds for all the solutions to problems (P_λ^s) for any $0 < \lambda \le \Lambda$.

Theorem 3. *There exists a positive constant $C = C(N, s, \Omega, r, q)$ such that any solution u_λ to problem (P_λ^s) with $\frac{1}{2} < s < 1$, $N > 2s$, $0 < q < 1 < r < \frac{N+2s}{N-2s}$ and $\lambda \in (0, \Lambda]$ satisfies*

$$\sup_{x\in\Omega} u_\lambda(x) \le C\,.$$

We also obtain uniform L^∞-estimates, in the case in which we move the boundary condition. As done in Chapter 1, we consider a family of sets $\{\Sigma_{\mathcal{D}}(\alpha)\}$, with $\alpha \in (0, |\partial\Omega|]$ satisfying the hypotheses (B_1)-(B_3) above. We set $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega \backslash \Sigma_{\mathcal{D}}(\alpha)$ and we assume that $\Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha) = \Gamma(\alpha)$ is a $(N-2)$-dimensional smooth submanifold. For a family of this type we consider the corresponding family of mixed boundary value problems,

$$(P_{\alpha,\lambda}) \qquad \left\{ \begin{array}{ll} (-\Delta)^s u = \lambda u^q + u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{array} \right.$$

where $B_\alpha(u)$ is defined as $B(u)$ with $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ replaced by $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$ satisfying the corresponding hypotheses $(\mathfrak{B}_\alpha)$. Under these conditions we prove the following.

Theorem 4. *Consider the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in(0,|\partial\Omega|]}$ satisfying the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3). For every $0 < \varepsilon < |\partial\Omega|$, let us denote $I_\varepsilon = [\varepsilon, |\partial\Omega|]$ and let*

$$\mathcal{S}_\varepsilon = \{u : \Omega \to \mathbb{R} |\ \textit{such that } u \textit{ is solution of } (P_{\alpha,\lambda}),\ \textit{with } \alpha \in I_\varepsilon\}.$$

Then, there exists a constant $\mathcal{M}_\varepsilon > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le \mathcal{M}_\varepsilon, \quad \forall\, u \in \mathcal{S}_\varepsilon.$$

In addition, we will also prove the following result about the behavior for the minimal solutions as we move the boundary condition:

Theorem 5. *Consider the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in(0,|\partial\Omega|]}$ satisfying the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3). Then*

(1) the minimal solutions $\{\underline{u}(\alpha)\}$ are uniformly bounded for any $\alpha \in [0, |\partial\Omega|]$. Moreover,

$$\|\underline{u}(\alpha)\|_{H^s(\Omega)},\ \|\underline{u}(\alpha)\|_{L^\infty(\Omega)} \to 0 \ \textit{as } \alpha \to 0;$$

(2) the non minimal solutions (of mountain pass type) are bounded and they converge to zero in $H^s(\Omega)$ as $\alpha \to 0$.

Chapter 2 is organized as follows: In Section 2.2 we introduce the appropriate functional framework. As performed in Chapter 1, using the ideas of [**24, 27, 29**], we also introduce an auxiliary problem that will help us to overcome some difficulties that appear when we deal with the fractional operator $(-\Delta)^s$. In Section 2.3 we study a half-space problem that will be useful to obtain the L^∞ bounds. To this end we make use of the moving planes method and we extend some results of [**37**] to the fractional setting. Section 2.4 is devoted to the study of the concave-convex problem by means of certain limit problems. This section contains the proof of Theorem 3 and Theorem 4 which are based on the blow-up method of [**60**]. To accomplish this step we need some compactness properties that requires to know precise Hölder estimates for the solutions to mixed Dirichlet-Neumann boundary problems. We use the results of Chapter 1 where the Hölder regularity of such solutions is proven. Section 2.5 is devoted to the proof of Theorem 2 and Theorem 5.

Our study of critical problems starts with **Part 2**, which is composed by the results of **Chapter 3**. In this chapter, we study the existence of solutions to the critical Brezis-Nirenberg problem when one deals with the spectral fractional Laplace operator and mixed Dirichlet-Neumann boundary condition,

$$(P_\lambda^c) \qquad \left\{\begin{array}{ll} (-\Delta)^s u = \lambda u + u^{2_s^*-1} & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{array}\right.$$

where $\frac{1}{2} < s < 1$, $2_s^* = \frac{2N}{N-2s}$ is the critical fractional Sobolev exponent, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and

$$B(u) = \chi_{\Sigma_{\mathcal{D}}} \cdot u + \chi_{\Sigma_{\mathcal{N}}} \cdot \frac{\partial u}{\partial \nu},$$

with $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ satisfying hypotheses $(\mathfrak{B})$ above. In the fractional setting, Brezis-Nirenberg problems have been widely investigated. For brevity we just cite some related works, e.g. [**18, 84**] for the spectral fractional Laplacian, and [**69, 79**] for the fractional Laplacian defined by a singular integral (see (1.2.3)); both with Dirichlet boundary condition. Up to our knowledge, there are no references dealing with the Brezis-Nirenberg problem involving the spectral fractional Laplacian and mixed Dirichlet-Neumann boundary condition. Subsequently, the main goal of this chapter is to address for the very first time the Brezis-Nirenberg problem in this fractional setting with mixed boundary conditions. Thus, the main result to be proven in this chapter is the following.

THEOREM 6. *Assume that $\frac{1}{2} < s < 1$ and $N \geq 4s$. Let $\lambda_{1,s}$ be the first eigenvalue of the fractional operator $(-\Delta)^s$ with homogeneous mixed Dirichlet-Neumann boundary condition. Then problem (P_λ^c)*

(1) has no solution for $\lambda \geq \lambda_{1,s}$,
(2) has at least one solution for $0 < \lambda < \lambda_{1,s}$,
(3) has at least one solution for $\lambda = 0$ and $\alpha = |\Sigma_{\mathcal{D}}|$ small enough.

Chapter 3 is organized as follows: First we introduce the appropriate functional space and, using the ideas of [**2**] and [**61**], we also introduce two constants $\widetilde{S}(\Sigma_{\mathcal{D}})$ and $\widetilde{S}(\Sigma_{\mathcal{N}})$ respectively, that play a similar role to that of the Sobolev constant in the classical Brezis-Nirenberg problem [**26**]. In Section 3.3 we study some useful properties of those constants. Section 3.4 is devoted to prove Theorem 6 and it is divided into two subsections. In Subsection 3.4.1 we prove the statements (1)-(2) of Theorem 6. In Subsection 3.4.2, we use the constant $\widetilde{S}(\Sigma_{\mathcal{D}})$ to study the existence of solutions of problem (P_λ^c) when we move the boundary condition as specified by hypotheses (B_1)-(B_3) above. This will allow us to prove statement (3) of Theorem 6. We conclude this chapter studying the nonlinear problem,

$$(P_f^s) \qquad \left\{\begin{array}{ll} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega. \end{array}\right.$$

Using a Pohozaev-type identity (see details in Section 3.5), we will be able to prove the following non-existence result.

THEOREM 7. *Assume that* u *is a solution of problem* (P_f^s) *and* f *is a continuous function with primitive* F. *Moreover, suppose that there exists* $x_0 \in \Omega$ *such that* $\langle x - x_0, \nu \rangle = 0$ *on* $\Sigma_{\mathcal{N}}$ *and* $\langle x - x_0, \nu \rangle > 0$ *on* $\Sigma_{\mathcal{D}}$. *If* f *and* F *satisfy the inequality* $(N - 2s)tf(t) - 2NF(t) \geq 0$, *then problem* (P_f^s) *has no solution.*

These important results highlight a big difference between a mixed boundary condition problem and a Dirichlet one as well as the relevance of the geometry of Ω and the decomposition of $\partial\Omega$ into $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ in the existence issues. In the Dirichlet case, if Ω is a star-shaped domain, then problem (P_f^s) has no solution under the growth condition for the nonlinearity given in Theorem 7. In particular, for the pure critical power case $f(t) = t^{2_s^*-1}$, it follows that the critical problem (P_0^c) has no solution under homogeneous Dirichlet boundary data and the star-shapeness assumption on Ω. On the other hand, in the mixed Dirichlet-Neumann boundary data case the situation is different and letting $\alpha = |\Sigma_{\mathcal{D}}|$ small enough, the existence of solution of problem (P_0^c) is guaranteed because of Theorem 6. Those aspects will be discussed in detail throughout this chapter.

Finally, we devote **Part 3** to the study of some nonlinear elliptic problems involving inverse operators and Dirichlet boundary conditions. Our interest in this kind of problems starts with the study of a fourth-order differential equation with homogeneous Navier boundary conditions and a nonlinear term depending on a second order differential operator, namely,

$$(P_\lambda^2) \qquad \left\{ \begin{array}{ll} (-\Delta)^2 u = \lambda u + (-\Delta)|u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega. \end{array} \right.$$

In particular, positive solutions of problem (P_λ^2) can be seen as positive steady-state solutions of the *fourth-order parabolic Cahn–Hilliard type-equation*,

$$\frac{\partial u}{\partial t} + (-\Delta)^2 u = \gamma u + (-\Delta)|u|^{p-1}u, \quad \text{in} \quad \Omega \times \mathbb{R}_+.$$

In **Chapter 4** we focus on existence issues for a problem closely related to (P_λ^2). More precisely, we study a second order equation involving a nonlocal term and Dirichlet boundary conditions,

$$(P_\lambda) \qquad \left\{ \begin{array}{ll} -\Delta u = \lambda(-\Delta)^{-1}u + |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

and we prove the existence of positive solutions for problem (P_λ) in any dimension $N > 6$ depending on the real parameter $\lambda > 0$, up to the critical value for the exponent p, i.e., when $1 < p \leq 2^* - 1$, where $2^* = \frac{2N}{N-2}$ is the critical exponent for the classical Sobolev Embedding Theorem.

In addition, we have a connection between problem (P_λ^2) and a second order elliptic system through problem (P_λ). Indeed, taking $v := \sqrt{\lambda}(-\Delta)^{-1}u$, problem (P_λ) provides us with the variational system,

$$(S_\lambda) \qquad \begin{cases} -\Delta u = \sqrt{\lambda} v + |u|^{p-1}u, \\ -\Delta v = \sqrt{\lambda} u, \end{cases} \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{in} \quad \partial\Omega,$$

which gives a different perspective to the problem in hand. In fact, we shall obtain the main results of this chapter following both perspectives with respect to the nonlocal equation (P_λ) and to the second order elliptic system (S_λ).
Let us observe that, at the critical exponent $p = 2^* - 1$, problem (P_λ) can be seen as a linear perturbation of the critical problem,

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega \subset \mathbb{R}^N, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

for which, after applying the well-known result of Pohozaev [**75**], one can prove the non-existence of positive solutions under the star-shapeness assumption on the domain Ω. Moreover, the classical Brezis-Nirenberg problem,

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \subset \mathbb{R}^N, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.3}$$

can also be seen as a linear perturbation of problem (0.2) so that the nonlocal term $\lambda(-\Delta)^{-1}u$ plays actually the role of λu in (0.3).
The main results proven in this chapter are the following.

Theorem 8. *Assume $1 < p < 2^* - 1$ and let $\lambda_{1,2}$ be the first eigenvalue of the operator $(-\Delta)^2$ with homogeneous Navier boundary conditions. Then, for every $\lambda \in (0, \lambda_{1,2})$ there exists at least a positive solution u of problem (P_λ).*

Theorem 9. *Assume $p = 2^* - 1$ and let $\lambda_{1,2}$ be the first eigenvalue of the operator $(-\Delta)^2$ with homogeneous Navier boundary conditions. Then, for every $\lambda \in (0, \lambda_{1,2})$, there exists at least a positive solution u of problem (P_λ) provided $N > 6$.*

Surprisingly, even though problem (P_λ) is a nonlocal but also a linear perturbation of the problem (0.2), Theorem 9 addresses dimensions $N > 6$, in contrast to the existence result by Brezis and Nirenberg (see [**26**]) about the linear perturbation (0.3), that covers the wider range $N \geq 4$. Despite of being just a linear perturbation, the nonlocal term $\lambda(-\Delta)^{-1}u$ has an important effect on the dimensions for which the classical Brezis-Nirenberg technique based on the minimizers of the Sobolev constant works. We study this phenomena in detail throughout Chapter 4.
Finally, although the equivalence between the system (S_λ) and the nonlocal problem (P_λ) provides us with existence results for the system (S_λ) by means of Theorem 8 and Theorem 9, we prove independently the following.

Theorem 10. *Assume $1 < p < 2^* - 1$ and let $\lambda_{1,2}$ be the first eigenvalue of the operator $(-\Delta)^2$ with homogeneous Navier boundary conditions. Then, for every $\lambda \in (0, \lambda_{1,2})$, there exists at least a positive solution (u, v) to system (S_λ).*

Theorem 11. *Assume $p = 2^* - 1$ and let $\lambda_{1,2}$ be the first eigenvalue of the operator $(-\Delta)^2$ with homogeneous Navier boundary conditions. Then, for every $\lambda \in (0, \lambda_{1,2})$, there exists at least a positive solution (u, v) to system (S_λ) provided $N > 6$.*

Chapter 4 is organized as follows: First we study the interval of the parameter λ compatible with the existence of positive solutions, proving the necessary condition $0 < \lambda < \lambda_{1,2}$, where $\lambda_{1,2}$ is the first eigenvalue of the operator $(-\Delta)^2$ under homogeneous Navier boundary conditions. Next in Section 4.2, using the well-known Mountain Pass Theorem [**12**], we show that for the range $2 < p+1 \leq 2^*$ and $0 < \lambda < \lambda_{1,2}$ there actually exists at least a positive solution of problem (P_λ). If $2 < p+1 < 2^*$ one might apply the Mountain Pass Theorem directly since, as we will show, our problem possesses the mountain pass geometry and, thanks to the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ for $2 \leq p+1 < 2^*$, the Palais-Smale condition is satisfied. On the other hand, at the critical exponent $p = 2^*-1$, the compactness of the Sobolev embedding is lost and check whether the PS condition is satisfied becomes a delicate issue to solve. To overcome this lack of compactness we apply a concentration-compactness argument based on the Concentration-Compactness Principle due to P. L. Lions, c.f. [**65**], which allows us to prove the required Palais-Smale condition (see details in Section 4.2.1). We prove the results for problem (P_λ) in Section 4.2 and using similar ideas, for system (S_λ) in Section 4.3. We conclude Chapter 4 with Section 4.4, where we extend our study and, given an integer $m > 1$, we prove under similar hypotheses above that there exists at least a positive solution to the problem

$$\left\{\begin{array}{lr} -\Delta u = (-\Delta)^{-m}\lambda u + |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ \quad u = 0 & \text{on } \partial\Omega. \end{array}\right.$$

Due to the lack of a comparison principle for a higher order equations, to obtain the existence results we can not address this problem directly, and we need to use a similar correspondence to the one performed above for the problem (P_λ^2), now with an elliptic system of $m+1$ equations.

In **Chapter 5** we extend the results contained in Chapter 4 to the fractional framework. As a natural generalization of problem (P_λ^2), we consider the fractional elliptic problem,

$$(P_\lambda^\alpha) \qquad \left\{\begin{array}{lr} (-\Delta)^\alpha u = \lambda u + (-\Delta)^\beta |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ (-\Delta)^j u = 0, \ \text{for } 0 \leq j < \lfloor \alpha \rfloor & \text{on } \partial\Omega, \end{array}\right.$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \beta < 1$, $\beta < \alpha < 1+\beta$ and $\lambda > 0$.
Following the ideas of Chapter 4, we study the existence of positive solutions of a fractional elliptic problem involving an inverse fractional operator and a nonlinear term derived from (P_λ^α), namely,

$$(P_\lambda^{\alpha,\beta}) \qquad \left\{\begin{array}{lll} (-\Delta)^{\alpha-\beta} u = \lambda(-\Delta)^{-\beta}u + |u|^{p-1}u & \text{in} & \Omega, \\ \qquad u = 0 & \text{on} & \partial\Omega. \end{array}\right.$$

We will prove that for the subcritical range $1 < p < 2_\mu^*-1$, where $0 < \mu := \alpha - \beta < 1$ and $2_\mu^* = \frac{2N}{N-2\mu}$ is the critical exponent of the fractional Sobolev embedding, there exists at least a positive solution if $\lambda < \lambda_{1,\alpha}$, denoting $\lambda_{1,\alpha}$ as the first eigenvalue of the fractional Laplace operator $(-\Delta)^\alpha$ under homogeneous Navier boundary conditions. On the other hand, for the critical exponent case $p = 2_\mu^*-1$, we will show that there exists at least a positive solution if $\lambda < \lambda_{1,\alpha}$ and $N > 4\alpha - 2\beta$. The results of this section clarify the effect, previously manifested in Chapter 4, that the nonlocal term $(-\Delta)^{-\beta}$ has on the dimensions for which the technique based on the minimizers of the fractional Sobolev constant works.

Dealing with problem $(P_\lambda^{\alpha,\beta})$ presents some difficulties besides those that could naturally appear when we consider the critical exponent $p = 2_\mu^*-1$. Namely, to handle the inverse term

$(-\Delta)^{-\beta}$ in addition to the typical difficulties that arise when working with fractional operators. Following the sketch performed above in Chapter 4, we use the equivalence between problem $(P_\lambda^{\alpha,\beta})$ and a certain fractional elliptic variational system to surpass the difficulties that arise while working with the inverse fractional Laplace operator $(-\Delta)^{-\beta}$. In particular, this approach will help us to avoid ascertaining explicit estimations for this inverse term. Indeed, taking $\omega := (-\Delta)^{-\beta}u$, problem $(P_\lambda^{\alpha,\beta})$ provides us with the fractional elliptic cooperative system,

$$\left\{\begin{array}{l}(-\Delta)^{\mu}u=\lambda\omega+|u|^{p-1}u,\\ (-\Delta)^{\beta}\omega=u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,\omega)=(0,0) \quad \text{in} \quad \partial\Omega.$$

However, the above system is not a variational system. In order to obtain a variational system from problem $(P_\lambda^{\alpha,\beta})$ we use a similar idea to the one performed above for problem (P_λ), distinguishing now whether $\alpha = 2\beta$ or $\alpha \neq 2\beta$. In the first case we split the parameter λ equally. Let us say, we take $v := \sqrt{\lambda}\,\omega$ and, recalling that $\mu := \alpha-\beta$, we obtain the following fractional elliptic cooperative system,

$$(S_\lambda^{\beta}) \qquad \left\{\begin{array}{l}(-\Delta)^{\beta}u=\sqrt{\lambda}v+|u|^{p-1}u,\\ (-\Delta)^{\beta}v=\sqrt{\lambda}u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{on} \quad \partial\Omega.$$

In the second case, $\alpha \neq 2\beta$, we split the parameter λ as follows,

$$\left\{\begin{array}{l}(-\Delta)^{\mu}u=\lambda^{1-\beta/\alpha}v+|u|^{p-1}u,\\ (-\Delta)^{\beta}v=\lambda^{\beta/\alpha}u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{on} \quad \partial\Omega.$$

Since the above system is still not variational, we transform it into the following variational system,

$$(S_\lambda^{\alpha,\beta}) \qquad \left\{\begin{array}{r}\frac{1}{\lambda^{1-\beta/\alpha}}(-\Delta)^{\mu}u=v+\frac{1}{\lambda^{1-\beta/\alpha}}|u|^{p-1}u,\\ \frac{1}{\lambda^{\beta/\alpha}}(-\Delta)^{\beta}v=u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{on} \quad \partial\Omega.$$

Analogously to what happened with problem (P_λ) studied in Chapter 4, the problem $(P_\lambda^{\alpha,\beta})$ can be seen as a linear perturbation of the critical problem,

$$(0.4) \qquad \left\{\begin{array}{rll}(-\Delta)^{\mu}u=|u|^{2_\mu^*-2}u & \text{in} & \Omega,\\ u=0 & \text{on} & \partial\Omega,\end{array}\right.$$

for which, after applying a Pohozaev-type result [**24**, Proposition 5.5], one can prove the non-existence of positive solutions under the star-shapeness assumption on the domain Ω. Moreover, the limit case $\beta \to 0$ in problem $(P_\lambda^{\alpha,\beta})$, provides us with the problem

$$(0.5) \qquad \left\{\begin{array}{rl}(-\Delta)^{\alpha}u=\lambda u+|u|^{2_\alpha^*-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{array}\right. \quad \text{with} \quad 0<\alpha<1,$$

which is analyzed in [**18**] where the authors proved the existence of positive solutions for $N \geq 4\alpha$ if and only if $0 < \lambda < \lambda_{1,\alpha}$, with $\lambda_{1,\alpha}$ being first eigenvalue of the $(-\Delta)^\alpha$ operator under homogeneous Dirichlet boundary condition. Note that in our situation the nonlocal term $\lambda(-\Delta)^{-\beta}u = \gamma v$ plays actually the role of λu in [**18**].
The main results to be proven in Chapter 5 are the following.

THEOREM 12. *Assume $1 < p < 2_\mu^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,\alpha})$, where $\lambda_{1,\alpha}$ is the first eigenvalue of $(-\Delta)^\alpha$ under homogeneous Navier boundary conditions, there exists at least a positive solution of the problem $(P_\lambda^{\alpha,\beta})$.*

THEOREM 13. *Assume $p = 2^*_\mu - 1$. Then, for every $\lambda \in (0, \lambda_{1,\alpha})$, where $\lambda_{1,\alpha}$ is the first eigenvalue of $(-\Delta)^\alpha$ under homogeneous Navier boundary conditions, there exists at least a positive solution of the problem $(P_\lambda^{\alpha,\beta})$ provided that $N > 4\alpha - 2\beta$.*

Let us remark that Theorem 13 addresses dimensions $N > 4\alpha - 2\beta$, in contrast to the existence result [**18**, Theorem 1.2] about the linear perturbation (0.5), that covers the wider range $N \geq 4\alpha$. This phenomena was already manifested in problem (P_λ) studied in Chapter 4. Now, Theorem 13 shows clearly the influence that the inverse term $(-\Delta)^{-\beta}$ has on the dimension and the existence issues.
The organization of Chapter 5 mimics that of Chapter 4. First we study the interval of the parameter λ compatible with the existence of positive solutions, proving the necessary condition $0 < \lambda < \lambda_{1,\alpha}$, where $\lambda_{1,\alpha}$ is the first eigenvalue of the operator $(-\Delta)^\alpha$ under homogeneous Navier boundary conditions. Next, using the Mountain Pass Theorem, we prove that for the range $2 < p+1 \leq 2^*$ and $0 < \lambda < \lambda_{1,\alpha}$ there actually exists at least a positive solution for problem $(P_\lambda^{\alpha,\beta})$. If $1 < p+1 < 2^*_\mu$ one might apply the Mountain Pass Theorem directly since, as we will show, our problem possesses the mountain pass geometry and, thanks to the compactness of the Sobolev embedding $H_0^\mu(\Omega) \hookrightarrow L^{p+1}(\Omega),\ 2 \leq p+1 < 2^*_\mu$, the Palais-Smale condition is satisfied (see details in Section 5.2). However, at the critical exponent $p = 2^*_\mu - 1$, the compactness of the Sobolev embedding is lost and the problem becomes very delicate. To overcome this lack of compactness we apply a concentration-compactness argument relying on [**18**, Theorem 5.1], which is an adaptation to the fractional setting of the classical result of P.L. Lions, [**65**]. Then we will be able to prove that, under certain hypotheses, the Palais-Smale condition is satisfied (see details in Section 5.3).

We conclude this PhD Thesis dissertation with **Chapter 6**. Motivated by the high-order problems that originate studies of Chapter 4 and Chapter 5, in this final chapter we perform an homotopy analysis of a high-order problem in divergence form. In particular, we study the Cauchy Problem for a quasilinear degenerate high-order parabolic problem of the form

$$(P_{HG}) \qquad \begin{cases} u_t = (-1)^{m-1}\nabla \cdot (f^n(|u|)\nabla \Delta^{m-1} u) & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

with $m \in \mathbb{N},\ m > 1$ and $n > 0$ is a fixed exponent, f is a continuous monotone increasing positive bounded function with $f(0) = 0$ and the initial data $u_0(x)$ is a bounded smooth compactly supported function.
The principal issue to overcome will be to detect proper solutions of the Cauchy Problem for the degenerate problem (P_{HG}) by uniformly parabolic analytic ε-regularizations. To this end, using the ideas of [**8**], we use an analytic homotopy approach based on *a priori* estimates for solutions of uniformly parabolic analytic ε-regularization problems, namely

$$\begin{cases} u_t = (-1)^{m-1}\nabla \cdot (\phi_\varepsilon(u)\nabla \Delta^{m-1} u) & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{0.6}$$

where $\phi_\varepsilon(u)$, with $\varepsilon \in (0,1]$, is an analytic ε-regularization such that $\phi_0(u) = f^n(|u|)$ and $\phi_1(u) = 1$. These *a priori* estimates will be obtained using classic techniques relying on integral identities for weak solutions.
Next, we study an analytic homotopy transformation in both parameters, $\varepsilon \to 0^+$ and $n \to 0^+$

and describe branching of solutions of problem (P_{HG}) from the polyharmonic heat equation

$$
\left\{ \begin{array}{ll} u_t = -(-\Delta)^m u & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{array} \right. \tag{0.7}
$$

which provides some qualitative oscillatory properties of solutions to (P_{HG}), at least for small $n > 0$.

As we will see, due to the similarity of the expressions for weak solutions of the Cauchy Problem (P_{HG}) and the Free Boundary Problem corresponding to the evolution of the support of the solution of (P_{HG}) the previous analysis based on ε-regularizations is unable to distinguish both type of solutions. Another issue that we will be unable to solve, due to the nature of the term $f(|u|)$, is the uniqueness of the limit of $u_\varepsilon(x,t)$ as $\varepsilon \to 0^+$. In the case $f(t) = t$, thanks to the scaling properties of $f(t)$, this problem is studied with an affirmative conclusion; see [**8**]. Also, we can not discard the dependence of the solution from the type of analytic ε-regularization $\phi_\varepsilon(u)$. Hence, we must carry out alternative arguments which could solve some of the issues explained above.

Subsequently, after this limit procedure in the ε-regularization we perform a second limit as $n \to 0^+$. That is, a continuous connection with solutions to the polyharmonic heat equation (0.7). Finally, we perform a double limit $n, \varepsilon \to 0^+$ from which we obtain the conditions on the parameters ε and n needed to obtain such a functional convergence. Now we state the main result of this final chapter.

Theorem 14. *Suppose that*

$$
n|\ln f(\varepsilon(n))| \to 0, \quad \textit{as } n \to 0^+,
$$

and the regularization family $\{u_\varepsilon(x,t)\}_{\varepsilon \in (0,1]}$ is uniformly bounded. Then

(1) *The solution $u(x,t)$ of the regularized problem*

$$
\left\{ \begin{array}{ll} u_t = (-1)^{m-1} \nabla \cdot \left(f^n \left((\varepsilon^2 + u^2)^{1/2}\right) \nabla \Delta^{m-1} u\right) & \textit{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \textit{in } \mathbb{R}^N, \end{array} \right.
$$

converges uniformly to the solution $u_{PH}(x,t)$ of the polyharmonic heat equation (0.7) *as $n \to 0^+$ and $\varepsilon \to 0^+$.*

(2) *If the convolution*

$$
\varphi(x,t) = -\int_0^t \nabla H(x, t-s) * \ln |u_{PH}(x,s)| \nabla \Delta^{m-1} u_{PH}(x,s) ds,
$$

remains bounded for the solution of the polyharmonic heat equation (0.7)*, the rate of convergence as $n \to 0^+$ of the asymptotic expansion $u(x,t) = u_{PH}(x,t) + V$ is given by*

$$
V := n\varphi + o(n).
$$

Extending the homotopic argument explained above for a fourth-order equation with an extra nonlinear term depending on a second order as $(-\Delta)|u|^{p-1}u$ could provide us with additional information about families of solutions of Cahn-Hilliard equations similar to those analysed in Chapter 4.

Part 1

Subcritical Problems with Mixed Dirichlet-Neumann Boundary data

CHAPTER 1

Regularity of solutions of a linear fractional elliptic problem with mixed Dirichlet-Neumann boundary conditions

This first chapter is devoted to the study of some regularity properties of solutions to linear fractional elliptic problems with mixed Dirichlet-Neumann boundary data when dealing with the Spectral Fractional Laplacian.

1.1. Introduction

In this chapter we study some regularity properties of the solutions to linear fractional elliptic problems such as

$$(P^s) \qquad \begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$, $p > \frac{N}{2s}$ and Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$. By $B(u)$ we mean the mixed Dirichlet-Neumann boundary condition, i.e.

$$B(u) = \chi_{\Sigma_{\mathcal{D}}}(x) \cdot u + \chi_{\Sigma_{\mathcal{N}}}(x) \cdot \frac{\partial u}{\partial \nu},$$

where ν is the outwards normal to $\partial\Omega$, χ_A stands for the characteristic function of the set A and $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ satisfy the hypotheses:

$$(\mathfrak{B}) \quad \begin{cases} \Sigma_{\mathcal{D}} \text{ and } \Sigma_{\mathcal{N}} \text{ are smooth } (N-1)\text{-dimensional submanifolds of } \partial\Omega. \\ \Sigma_{\mathcal{D}} \text{ is a closed manifold of positive } (N-1)\text{-dimensional Lebesgue measure,} \\ \qquad\qquad |\Sigma_{\mathcal{D}}| = \alpha \in (0, |\partial\Omega|). \\ \Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset\,,\ \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega \text{ and } \Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma \text{ where } \Gamma \text{ is a smooth} \\ (N-2)\text{-dimensional submanifold of } \partial\Omega. \end{cases}$$

The main result we prove in this chapter is the following.

Theorem 1.1.1. *Assume that Ω is a smooth bounded domain of $\mathbb{R}^N$ such that $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ satisfy the hypotheses $(\mathfrak{B})$ and let u be the solution to problem (P^s) with $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. Then $u \in \mathcal{C}^\gamma(\overline{\Omega})$ for some $0 < \gamma < \frac{1}{2}$. Even more, there exists a constant $\mathscr{H} = \mathscr{H}(N, s, f, p, |\Sigma_{\mathcal{D}}|) > 0$ such that*

$$|u(x) - u(y)| \leq \mathscr{H}|x-y|^\gamma,\ \forall\ x, y \in \overline{\Omega}.$$

To prove Theorem 1.1.1 we follow some of the ideas in [**62, 80**]. Using the De Giorgi truncation method, Stampacchia (see [**80**]) established the regularity of solutions to the mixed boundary problem involving the classical Laplace operator. Due to the nonlocal nature of problem (P^s), some difficulties arise when trying to apply this truncation method to solutions to (P^s). Based on the ideas of [**29, 27, 24**], at this point we will make full use of the local realization of the fractional operator $(-\Delta)^s$ in terms of certain auxiliary degenerate elliptic problem. We use the results of [**48**] to adapt the procedures of [**80**] to the case of degenerate elliptic equations with weights in the Muckenhoupt class A_2 (we refer to [**48**] for the precise definition as well as some useful properties of those weights).

In addition to Theorem 1.1.1, following some ideas in [**34**], in the last part of this chapter we study the behavior of the problem (P^s) when we move the boundary condition in a regular way as specified next.

Given $I_\varepsilon = [\varepsilon, |\partial\Omega|]$ for some $\varepsilon > 0$, let us consider the family of closed sets $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$, satisfying the hypotheses:

(B_1) $\Sigma_{\mathcal{D}}(\alpha)$ has a finite number of connected components.
(B_2) $\Sigma_{\mathcal{D}}(\alpha_1) \subset \Sigma_{\mathcal{D}}(\alpha_2)$ if $\alpha_1 < \alpha_2$.
(B_3) $|\Sigma_{\mathcal{D}}(\alpha_1)| = \alpha_1 \in I_\varepsilon$.

We denote by $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega \backslash \Sigma_{\mathcal{D}}(\alpha)$ and $\Gamma(\alpha) = \Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha)$. For a family of this type we consider the corresponding family of mixed boundary value problems,

$$(P^s_\alpha) \qquad \left\{ \begin{array}{ll} (-\Delta)^s u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{array} \right.$$

where $B_\alpha(u)$ is the boundary condition associated to the parameter α in the previous hypotheses and the boundary manifolds $\Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}}(\alpha)$ satisfy the corresponding hypotheses $(\mathfrak{B}_\alpha)$. In this scenario we prove the following result.

Corollary 1.1.1. *Given Ω a smooth bounded domain such that the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$ satisfies the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)–(B_3), let u_α be the solution to (P^s_α) with $\frac{1}{2} < s < 1$, $f \in L^p(\Omega)$ and $p > \frac{N}{2s}$. Then, there exist two constants $0 < \gamma < \frac{1}{2}$ and $\mathscr{H}_\varepsilon > 0$ both independent of $\alpha \in [\varepsilon, |\partial\Omega|]$ such that*

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\overline{\Omega})} \leq \mathscr{H}_\varepsilon.$$

As we will see in the proof of Corollary 1.1.1 , when one takes $\alpha \to 0^+$ the control of the Hölder norm of such family is lost. Hence, it is necessary to fix a positive minimum $\varepsilon > 0$ on the measure of the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$, in order to guarantee the control on the Hölder norm for the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$.

1.2. Functional setting and preliminaries

The definition of the fractional powers of the positive Laplace operator $(-\Delta)$, in a bounded domain Ω with homogeneous mixed Dirichlet-Neumann boundary data, is carried out via the spectral decomposition using the powers of the eigenvalues of $(-\Delta)$ with the same boundary condition. Let (φ_i, λ_i) be the eigenfunctions (normalized with respect to the $L^2(\Omega)$-norm) and eigenvalues of $(-\Delta)$ with homogeneous mixed Dirichlet-Neumann boundary data, then (φ_i, λ_i^s) are the eigenfunctions and eigenvalues of $(-\Delta)^s$ with the same boundary conditions. Hence, given $u_i(x) = \sum_{j\geq 1} \langle u_i, \varphi_j\rangle \varphi_j$, $i = 1, 2$

$$\langle (-\Delta)^s u_1, u_2\rangle = \sum_{j\geq 1} \lambda_j^s \langle u_1, \varphi_j\rangle \langle u_2, \varphi_j\rangle,$$

i.e., the action of the fractional operator on a function u is given by

$$(-\Delta)^s u = \sum_{j\geq 1} \lambda_j^s \langle u, \varphi_j\rangle \varphi_j.$$

Thus, the fractional Laplace operator $(-\Delta)^s$ is well defined in the space of functions that vanish on $\Sigma_{\mathcal{D}}$,

$$H^s_{\Sigma_{\mathcal{D}}}(\Omega) = \left\{ u = \sum_{j\geq 1} a_j\varphi_j \in L^2(\Omega) : \ \|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \sum_{j\geq 1} a_j^2 \lambda_j^s < \infty \right\}.$$

As a direct consequence of the previous definition, given $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$, it follows that

$$\|u\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}. \tag{1.2.1}$$

Next, let us recall some well-known facts about fractional Sobolev spaces. The fractional Sobolev space $H^s(\Omega)$, $0 < s < 1$, with $\Omega \subset \mathbb{R}^N$ a bounded domain is defined as the set of functions $u \in L^2(\Omega)$ such that the norm

$$\|u\|_{H^s(\Omega)} := \|u\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2},$$

is finite. Because of [**66**, Chapter 2], we can characterize the space $H^s_0(\Omega)$ obtained by considering classical Dirichlet boundary conditions (i.e.$\Sigma_{\mathcal{D}} = \partial\Omega$) as the completion of $\mathcal{C}^\infty_0(\Omega)$ under the norm $\|\cdot\|_{H^s(\Omega)}$. In particular, for $0 < s < 1$, $s \neq \frac{1}{2}$,

$$H^s_0(\Omega) = \overline{\mathcal{C}^\infty_0(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

Moreover, due to [**66**, Theorem 11.1], if $0 < s \leq \frac{1}{2}$ then $H^s_0(\Omega) = H^s(\Omega)$, while for $\frac{1}{2} < s < 1$ we have the strict inclusion $H^s_0(\Omega) \subsetneq H^s(\Omega)$. Since $H^s_0(\Omega) \subset H^s_{\Sigma_{\mathcal{D}}}(\Omega) \subset H^s(\Omega)$, it follows that $H^s_{\Sigma_{\mathcal{D}}}(\Omega) = H^s(\Omega)$ for $0 < s \leq \frac{1}{2}$. Hence, the range $\frac{1}{2} < s < 1$, for which we have $H^s_{\Sigma_{\mathcal{D}}}(\Omega) \subsetneq H^s(\Omega)$, provides us with the correct functional space to study the mixed boundary problem (P^s). On the other hand, this definition of the fractional powers of the Laplace operator allows us to integrate by parts in the appropriate spaces, so that a natural definition of weak solution of problem (P_s) is the following.

DEFINITION 1.2.1. *We say that* $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ *is a solution of problem* (P^s) *if*

$$\int_\Omega (-\Delta)^{s/2} u \, (-\Delta)^{s/2}\psi dx = \int_\Omega f\psi dx, \quad \textit{for all } \psi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega).$$

Due to the nonlocal nature of the fractional operator $(-\Delta)^s$ some difficulties arise when one tries to obtain explicit expressions involving the action of the fractional Laplacian on a given function. In order to overcome these difficulties, we use the ideas of Caffarelli and Silvestre, see [**29**], together with those of [**27, 24**] to give an equivalent definition of the operator $(-\Delta)^s$ by means of an auxiliary problem that we introduce next.
Given a domain Ω, we set the cylinder $\mathscr{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. We denote with (x, y) points that belong to $\mathscr{C}_\Omega$ and with $\partial_L\mathscr{C}_\Omega = \partial\Omega \times [0,\infty)$ the lateral boundary of the cylinder. Let us also denote with $\Sigma^*_{\mathcal{D}} = \Sigma_{\mathcal{D}} \times [0,\infty)$ and $\Sigma^*_{\mathcal{N}} = \Sigma_{\mathcal{N}} \times [0,\infty)$ as well as $\Gamma^* = \Gamma \times [0,\infty)$. It is clear that, by construction,

$$\Sigma^*_{\mathcal{D}} \cap \Sigma^*_{\mathcal{N}} = \emptyset\,, \quad \Sigma^*_{\mathcal{D}} \cup \Sigma^*_{\mathcal{N}} = \partial_L\mathscr{C}_\Omega \quad \text{and} \quad \Sigma^*_{\mathcal{D}} \cap \overline{\Sigma^*_{\mathcal{N}}} = \Gamma^*\,.$$

Given a function $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ we define its s-harmonic extension, denoted by $U = E_s[u]$, as the solution of the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(U) = 0 & \text{on } \partial_L \mathscr{C}_\Omega, \\ U(x,0) = u(x) & \text{on } \Omega \times \{y=0\}, \end{cases}$$

where

$$B(U) = \chi_{\Sigma^*_{\mathcal{D}}} \cdot U + \chi_{\Sigma^*_{\mathcal{N}}} \cdot \frac{\partial U}{\partial \nu},$$

being ν, with an abuse of notation[1], the outwards normal to $\partial_L \mathscr{C}_\Omega$. The extension function belongs to the space

$$\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega) := \overline{\mathcal{C}^\infty_0((\Omega \cup \Sigma_{\mathcal{N}}) \times [0,\infty))}^{\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}},$$

where

$$\|U\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} := \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla U(x,y)|^2 dxdy,$$

for $\kappa_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$. Note that $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ is a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}$ which is induced by the scalar product

$$\langle U, V\rangle_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} = \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla U, \nabla V\rangle dxdy.$$

Moreover, the following inclusions are satisfied,

$$\mathcal{X}^s_0(\mathscr{C}_\Omega) \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega) \subsetneq \mathcal{X}^s(\mathscr{C}_\Omega),$$

with $\mathcal{X}^s_0(\mathscr{C}_\Omega)$ the space of functions that belongs to $\mathcal{X}^s(\mathscr{C}_\Omega) \equiv H^1(\mathscr{C}_\Omega, y^{1-2s}dxdy)$ and vanish on the lateral boundary of $\mathscr{C}_\Omega$. Accordingly to [**29, 24**], due to the choice of the constant κ_s, the extension operator E_s is an isometry, i.e.

$$\|E_s[\varphi]\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} = \|\varphi\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}, \text{ for all } \varphi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{1.2.2}$$

The key point of the extension function is that it is related to the fractional Laplacian of the original function through the formula

$$\frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y\to 0^+} y^{1-2s} \frac{\partial w}{\partial y} = (-\Delta)^s u(x).$$

In the case $\Omega = \mathbb{R}^N$ this formulation provides us with explicit expressions for both the fractional Laplacian and the s-extension in terms of the Riesz and the Poisson kernels, respectively. Namely,

$$\begin{aligned} U(x,y) &= P^s_y * u(x) = c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{u(z)}{(|x-z|^2 + y^2)^{\frac{N+2s}{2}}} dz \\ (-\Delta)^s u(x) &= d_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}. \end{aligned} \tag{1.2.3}$$

We refer to [**24**] for the exact values of the constants κ_s, $c_{N,s}$ and $d_{N,s}$ as well as the existent relation between them, namely, $2s\kappa_s c_{N,s} = d_{N,s}$.

---

[1] Let ν be the outwards normal vector to $\partial\Omega$ and $\nu_{(x,y)}$ the outwards normal to $\mathscr{C}_\Omega$ then, by construction, $\nu_{(x,y)} = (\nu, 0)$, $y > 0$.

Using the above arguments we can reformulate the problem (P^s) in terms of the extension problem as follows:

$$(P_s^*) \qquad \left\{ \begin{aligned} -\mathrm{div}(y^{1-2s}\nabla U) &= 0 && \text{in } \mathscr{C}_\Omega, \\ B(U) &= 0 && \text{on } \partial_L \mathscr{C}_\Omega, \\ \frac{\partial U}{\partial \nu^s} &= f && \text{on } \Omega \times \{y=0\}. \end{aligned} \right.$$

DEFINITION 1.2.2. *An* energy solution *to problem* (P_s^*) *is a function* $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ *such that*

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla U, \nabla \varphi \rangle \, dxdy = \int_\Omega f(x)\varphi(x,0)dx, \qquad \forall \varphi \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega). \tag{1.2.4}$$

Given $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ a solution of problem (P_s^*) the function $u(x) = Tr[U(x,y)] = U(x,0)$ belongs to $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and solves problem (P^s). Moreover, also the vice versa is true: given a solution $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ of problem (P^s), its s-harmonic extension $U = E_s[u] \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ is a solution of (P_s^*). Thus, both formulations are equivalent and the *Extension operator*

$$E_s : H^s_{\Sigma_{\mathcal{D}}}(\Omega) \to \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega),$$

allows us to switch between both of them.

In the Dirichlet case, it is also proved in [**24**] that, given $z \in \mathcal{X}^s_0(\mathscr{C}_\Omega)$, there exists a constant $C = C(N, s, r, |\Omega|)$ such that the *trace inequality*,

$$\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla z(x,y)|^2 dxdy \geq C \left(\int_\Omega |z(x,0)|^r dx \right)^{\frac{2}{r}}, \tag{1.2.5}$$

holds provided $1 \leq r \leq 2_s^*,\ N > 2s$, with $2_s^* = \frac{2N}{N-2s}$. Such inequality turns out to be very useful and it is in fact equivalent to the fractional Sobolev inequality,

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 dx \geq C_1 \left(\int_\Omega |v|^r dx \right)^{\frac{2}{r}}, \qquad \forall v \in H^s_0(\Omega),\ 1 \leq r \leq 2_s^*,\ N > 2s. \tag{1.2.6}$$

REMARK 1.2.1. *When* $r = 2_s^*$ *the best constant in* (1.2.5) *will be denoted by* $S(s,N)$. *This constant is explicit and independent of the domain* Ω, *and its exact value is given by the following expression,*

$$S(s,N) = \frac{2\pi^s \Gamma(1-s)\Gamma(\frac{N+2s}{2})(\Gamma(\frac{N}{2}))^{\frac{2s}{N}}}{\Gamma(s)\Gamma(\frac{N-2s}{2})(\Gamma(N))^s}.$$

Since it is not achieved in any bounded domain (see Remarks 3.2.1-(1)) we have that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla z(x,y)|^2 dxdy \geq S(s,N) \left(\int_{\mathbb{R}^N} |z(x,0)|^{\frac{2N}{N-2s}} dx \right)^{\frac{N-2s}{N}}, \ z \in \mathcal{X}^s(\mathbb{R}^{N+1}_+).$$

Indeed, in the whole space case the latter inequality is achieved when $z = E_s[u]$ *and*

$$u(x) = u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}},$$

with arbitrary $\varepsilon > 0$, *cf.* [**24**]. *Finally, the best constant in* (1.2.6) *with* $\Omega = \mathbb{R}^N$ *is given by* $\kappa_s S(s,N)$.

When mixed boundary conditions are considered, the situation is quite similar since the Dirichlet condition is imposed on a set $\Sigma_{\mathcal{D}} \subset \partial\Omega$ such that $|\Sigma_{\mathcal{D}}| = \alpha > 0$. Hence, there exists a positive constant $C_{\mathcal{D}} = C_{\mathcal{D}}(N, s, |\Sigma_{\mathcal{D}}|)$ such that

$$0 < \inf_{\substack{u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega) \\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}} := C_{\mathcal{D}} < \inf_{\substack{u \in H^s_0(\Omega) \\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_0(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}}. \tag{1.2.7}$$

REMARK 1.2.2. *The constant $C_{\mathcal{D}}$ will be studied in detail throughout Chapter 3. Actually, we will prove that $C_{\mathcal{D}}(N, s, |\Sigma_{\mathcal{D}}|) \leq 2^{-\frac{2s}{N}} \kappa_s S(s, N)$, for $S(s, N)$ the best constant in* (1.2.5) *with $r = 2^*_s$ (see Proposition 3.3.2). Moreover, taking in mind the spectral definition of the fractional operator and making use of the Hölder inequality, it follows that $C_{\mathcal{D}} \leq |\Omega|^{\frac{2s}{N}} \lambda_1^s(\alpha)$, with $\lambda_1(\alpha)$ the first eigenvalue of the Laplace operator with mixed boundary conditions on the sets $\Sigma_{\mathcal{D}} = \Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}} = \Sigma_{\mathcal{N}}(\alpha)$. Since $\lambda_1(\alpha) \to 0$ as $\alpha \to 0^+$, see* [**34**, Lemma 4.3], *we conclude that $C_{\mathcal{D}} \to 0$ as $\alpha \to 0^+$.*

Gathering together (1.2.2) and (1.2.7), we find

$$C_{\mathcal{D}} \left(\int_\Omega \varphi^{2^*_s}(x, 0) dx \right)^{\frac{2}{2^*_s}} \leq \|\varphi(x, 0)\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} = \|E_s[\varphi(x, 0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}. \tag{1.2.8}$$

With this Sobolev-type inequality in hand we can prove a trace inequality adapted to our mixed boundary data framework.

LEMMA 1.2.1. *There exists a constant $C_{\mathcal{D}} = C_{\mathcal{D}}(N, s, |\Sigma_{\mathcal{D}}|) > 0$ such that,*

$$\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla \varphi|^2 dx dy \geq C_{\mathcal{D}} \left(\int_\Omega \varphi^{2^*_s}(x, 0) dx \right)^{\frac{2}{2^*_s}}, \qquad \forall \varphi \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega). \tag{1.2.9}$$

PROOF. Thanks to (1.2.8), it is enough to prove that $\|E_s[\varphi(\cdot, 0)]\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \leq \|\varphi\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}$. This inequality is satisfied since, arguing as in [**24**], we find

$$\begin{aligned}
\|\varphi\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} &:= \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla \varphi|^2 dx dy \\
&= \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla \left(E_s[\varphi(x, 0)] + \varphi(x, y) - E_s[\varphi(x, 0)] \right)|^2 dx dy \\
&= \|E_s[\varphi(x, 0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|\varphi(x, y) - E(\varphi(x, 0))\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \\
&\quad + 2\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla E_s[\varphi(x, 0)], \nabla(\varphi(x, y) - E_s[\varphi(x, 0)]) \rangle dx dy \\
&= \|E_s[\varphi(x, 0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|\varphi(x, y) - E_s[\varphi(x, 0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \\
&\quad + 2 \int_\Omega (-\Delta)^s(\varphi(x, 0))(\varphi(x, 0) - \varphi(x, 0)) dx \\
&= \|E_s[\varphi(x, 0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|\varphi(x, y) - E_s[\varphi(x, 0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}.
\end{aligned}$$

$\square$

1.3. Hölder Regularity

The principal result we will show in this section is Theorem 1.1.1, which deals with the Hölder regularity of the solution to problem (P^s). First we introduce the notation that we will follow along this section.

Notation. Given an open bounded set Ω, $x \in \overline{\Omega} \subset \mathbb{R}^N$ and $X \in \overline{\mathscr{C}}_\Omega \subset \mathbb{R}^{N+1}_+$, we define

- $\Omega(x,\rho) = \Omega \cap B_\rho(x)$,
- $\mathscr{C}_\Omega(X,\rho) = \mathscr{C}_\Omega \cap B_\rho(X)$,

Given $u(x) \in H^s_{\Sigma_\mathcal{D}}(\Omega)$ and $U(X) \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)$, let us also define

- $A_+(k) = \{x \in \Omega : u(x) > k\}$,
- $A^*_+(k) = \{X \in \mathscr{C}_\Omega : U(X) > k\}$,
- $A_+(k,\rho) = A_+(k) \cap \Omega(x,\rho)$
- $A^*_+(k,\rho) = A^*_+(k) \cap \mathscr{C}_\Omega(X,\rho)$,
- $\{\cdot\}^k = \min(\cdot, k)$.
- $\{\cdot\}_k = \max(\cdot, k)$.

In a similar way we may define the sets $A_-(k)$, $A^*_-(k)$, $A_-(k,\rho)$ and $A^*_-(k,\rho)$ replacing $>$ with $<$ in the latter definitions. We denote by

- $|A|_\omega$ the measure induced by a weight ω of the set A.
- $|A|_{y^{1-2s}}$ the measure induced by the weight y^{1-2s} of the set A.
- $|A|$ the usual Lebesgue measure of the set A.

Let $z \in \overline{\Omega}$ and $R > 0$. Given a solution u to problem (P^s), we write $u(x) = v(x) + w(x)$ for every $x \in \Omega(z,R)$, where the function $v(x)$ satisfies

$$\left\{\begin{aligned} (-\Delta)^s v &= f && \text{in } \Omega(z,R),\\ v &= 0 && \text{on } \widetilde{\Sigma}_{\mathcal{D},R} := \partial\Omega(z,R)\backslash\Sigma_\mathcal{N},\\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } \widetilde{\Sigma}_{\mathcal{N},R} := \partial\Omega(z,R)\cap\Sigma_\mathcal{N}, \end{aligned}\right. \tag{1.3.1}$$

and the function $w(x)$ is such that,

$$\left\{\begin{aligned} (-\Delta)^s w &= 0 && \text{in } \Omega(z,R),\\ w &= 0 && \text{on } \Sigma_{\mathcal{D},R} := \Sigma_\mathcal{D} \cap B_R(z),\\ \frac{\partial w}{\partial \nu} &= 0 && \text{on } \Sigma_{\mathcal{N},R} := \Sigma_\mathcal{N} \cap B_R(z), \end{aligned}\right. \tag{1.3.2}$$

Using the extension technique we can write $v(x) = V(x,0)$ with $V(x,y)$ a solution of the extended problem

$$\left\{\begin{aligned} -\text{div}\,(y^{1-2s}\nabla V) &= 0 && \text{in } \mathscr{C}_{\Omega(z,R)},\\ B(V) &= 0 && \text{on } \partial_L\mathscr{C}_{\Omega(z,R)},\\ \frac{\partial V}{\partial \nu^s} &= f && \text{on } \Omega(z,R)\times\{y=0\}, \end{aligned}\right. \tag{1.3.3}$$

where $B(V) = V\chi_{\widetilde{\Sigma}^*_{\mathcal{D},R}} + \dfrac{\partial V}{\partial \nu}\chi_{\widetilde{\Sigma}^*_{\mathcal{D},R}}$, with $\widetilde{\Sigma}^*_{\mathcal{D},R} = \widetilde{\Sigma}_{\mathcal{D},R}\times[0,\infty)$ and $\widetilde{\Sigma}^*_{\mathcal{N},R} = \widetilde{\Sigma}_{\mathcal{N},R}\times[0,\infty)$. In the same way, we write $w(x) = W(x,0)$, with $W(x,y)$ satisfying the extended problem

$$\left\{\begin{aligned} -\mathrm{div}(y^{1-2s}\nabla W) &= 0 && \text{in } \mathscr{C}_{\Omega(z,R)},\\ B(W) &= 0 && \text{on } \Sigma^*_{\mathcal{D},R}\cup\Sigma^*_{\mathcal{N},R},\\ \frac{\partial W}{\partial \nu^s} &= 0 && \text{on } \Omega(z,R)\times\{y=0\}, \end{aligned}\right. \tag{1.3.4}$$

where $B(V) = V\chi_{\Sigma^*_{\mathcal{D},R}} + \dfrac{\partial V}{\partial \nu}\chi_{\Sigma^*_{\mathcal{D},R}}$, with $\Sigma^*_{\mathcal{D},R} = \Sigma_{\mathcal{D},R}\times[0,\infty)$ and $\Sigma^*_{\mathcal{N},R} = \Sigma_{\mathcal{N},R}\times[0,\infty)$.
Let us observe the following:

(i) If $z\in\Omega$, there exists $R>0$ such that $\widetilde{\Sigma}_{\mathcal{D},R} = \partial\Omega(z,R)$ and $\Sigma_{\mathcal{D},R} = \Sigma_{\mathcal{N},R} = \emptyset$. Then, $v\in H_0^s(\Omega(z,R))$ and it is solution of a Dirichlet problem. Moreover, given $W = E_s[w]$ with w satisfying (1.3.2), $W\in\mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ and it satisfies

$$\int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}\langle\nabla W,\nabla\Phi\rangle\, dxdy = 0, \qquad \forall\Phi\in\mathcal{X}_0^s(\mathscr{C}_{\Omega(z,R)}). \tag{1.3.5}$$

(ii) If $z\in\Sigma_{\mathcal{D}}\backslash\Gamma$, there exists $R>0$ such that $\widetilde{\Sigma}_{\mathcal{D},R} = \partial\Omega(z,R)$ and $\Sigma_{\mathcal{N},R} = \emptyset$. Then, $v\in H_0^s(\Omega(z,R))$ and it is a solution of a Dirichlet problem; the extension function $W\in\mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ and (1.3.5) holds $\forall\Phi\in\mathcal{X}_0^s(\mathscr{C}_{\Omega(z,R)})$.

(iii) If $z\in\Sigma_{\mathcal{N}}$, there exists $R>0$ such that $\Sigma_{\mathcal{D},R} = \emptyset$. Then, the function $v\in H^s_{\widetilde{\Sigma}_{\mathcal{D},R}}(\Omega(z,R))$ and it is a solution of the mixed problem (1.3.1); the extension function $W\in\mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ and (1.3.5) holds $\forall\Phi\in\mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ vanishing on $\partial_L\mathscr{C}_{\Omega(z,R)}\backslash\Sigma^*_{\mathcal{N},R}$.

(iv) Finally, if $z\in\Gamma$, the sets $\widetilde{\Sigma}_{\mathcal{D},R}$, $\widetilde{\Sigma}_{\mathcal{N},R}$, $\Sigma_{\mathcal{D},R}$ and $\Sigma_{\mathcal{D},R}$ are nonempty for all $R>0$. Then, the function $v\in H^s_{\widetilde{\Sigma}_{\mathcal{D},R}}(\Omega(z,R))$ and it is a solution of the mixed problem (1.3.1); the extension function $W\in\mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ and (1.3.5) holds for all $\Phi\in\mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ vanishing on $\partial_L\mathscr{C}_{\Omega(z,R)}\backslash\Sigma^*_{\mathcal{N},R}$.

Accordingly to the above comments, we define

- $\mathscr{C}^\circ_{\Omega(z,R)} = \overline{\mathscr{C}}_{\Omega(z,R)}\backslash\{X=(x,y)\in\mathscr{C}_{\Omega(z,R)} : x\in\partial B_R(z)\}$,
- $\partial_0\mathscr{C}_{\Omega(z,R)} = \partial_L\mathscr{C}_{\Omega(z,R)}\backslash\Sigma^*_{\mathcal{N},R}$.
- $\partial_B\mathscr{C}_{\Omega(z,R)} = \partial_L\mathscr{C}_{\Omega(z,R)}\backslash\left(\Sigma^*_{\mathcal{D},R}\cup\Sigma^*_{\mathcal{N},R}\right)$.

We continue by stating the definitions and results needed in what follows. The first definition is based on [**80**, Definition 2.1].

Definition 1.3.1. *Given $Z\in\mathscr{C}^\circ_{\Omega(z,R)}$, let $\mathcal{K}^+(Z)$ (resp. $\mathcal{K}^-(Z)$) be the set of values $k\in\mathbb{R}$ such that there exists a number $\widetilde{\rho}(Z)>0$ satisfying $\{U\}^k\eta\in\mathcal{X}^s_{\partial_0\mathscr{C}_{\Omega(z,R)}}(\mathscr{C}_{\Omega(z,R)})$ (resp. $\{U\}_k\,\eta\in\mathcal{X}^s_{\partial_0\mathscr{C}_{\Omega(z,R)}}(\mathscr{C}_{\Omega(z,R)})$) for any $U\in\mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ and any function $\eta\in C^\infty(\mathbb{R}^{N+1}_+)$ such that $supp(\eta)\subset B_{\widetilde{\rho}(Z)}(Z)$.*

Let us observe that,

- If $Z\in\Sigma^*_{\mathcal{D},R}$ then $\mathcal{K}^+(Z) = [0,\infty)$, $\mathcal{K}^-(Z) = (-\infty,0]$ and $\widetilde{\rho}(Z) = dist(Z,\partial_B\mathscr{C}_{\Omega(z,R)})$.
- If $Z\in\mathscr{C}^\circ_{\Omega(z,R)}\backslash\Sigma^*_{\mathcal{D},R}$, then $\mathcal{K}^+(Z) = \mathcal{K}^-(Z) = (-\infty,\infty)$, and in this case $\widetilde{\rho}(Z) = dist(Z,\partial_0\mathscr{C}_{\Omega(z,R)})$.

– Because of the construction of the extension cylinder, it is immediate that the number $\widetilde{\rho}(Z) > 0$ does not depend on the y variable.

The control of the oscillations of solutions of elliptic problems is usually carried out through integral estimates that mainly rely on a Sobolev-type inequality. Since the extension function solves a degenerate elliptic problem involving the weight y^{1-2s} that belongs to the Muckenhoupt class A_2, it is necessary to establish a Sobolev-type inequality dealing with such a type of degenerate weights. To this aim, we recall the following results.

THEOREM 1.3.1 ([**48**], Theorem 1.3). *Let Ω be an open bounded set in $\mathbb{R}^N$ and consider $1 < p < \infty$ and a weight ω that belongs to the Muckenhoupt class A_p. Then, there exist a positive constant $C(\Omega)$ and $\delta > 0$ such that for all $u \in H_0^1(\Omega, \omega)$ and $1 \le \sigma \le \frac{N}{N-1} + \delta$ we have*

$$\|u\|_{L^{\sigma p}(\Omega,\omega dx)} \le C(\Omega)\|\nabla u\|_{L^p(\Omega,\omega dx)}, \tag{1.3.6}$$

where $C(\Omega) = c_\omega diam(\Omega)|\Omega|_\omega^{\frac{1}{p}\left(\frac{1}{\sigma}-1\right)}$ for a positive constant c_ω depending on N, p and ω.

THEOREM 1.3.2 ([**48**], Theorem 1.6). *Assume $1 < p < \infty$, $\omega \in A_p$ and suppose that there exists $0 < \xi < 1$ and $\rho_0 > 0$ such that for any $x_0 \in \partial\Omega$ and for any $0 < \rho < \rho_0$ we have*

$$|B_\rho(x_0)\backslash\Omega(x,\rho)| \ge \xi|B_\rho(x_0)|.$$

Then, there exist a positive constant $C = C(B_\rho(x_0))$ and $\delta > 0$ such that for any $x_0 \in \partial\Omega$, $1 \le \sigma \le \frac{N}{N-1} + \delta$ and any $u \in H^1(\overline{\Omega(x_0,\rho)}, \omega)$ vanishing on $\partial\Omega \cap B_\rho(x_0)$ we have

$$\|u\|_{L^{\sigma p}(\Omega(x_0,\rho),\omega dx)} \le C(B_\rho)\|\nabla u\|_{L^p((\Omega(x_0,\rho),\omega dx)},$$

where $C(\Omega) = c_\omega diam(\Omega)|\Omega|_\omega^{\frac{1}{p}\left(\frac{1}{\sigma}-1\right)}$ for a positive constant c_ω depending on ω, N, p and ξ.

According to [**48**], given $1 < p < \infty$ and a weight ω belonging to the Muckenhoupt class A_p, there exists $\epsilon_0 > 0$ such that $w \in A_q$ for all $q \ge p - \epsilon_0$. Then, for weights in the class A_2, we are allowed to use (1.3.6) with $p \ge 2 - \epsilon_0$ for some $\epsilon_0 > 0$. Let us notice that, in terms of the extension domains, such an inequality will involve domains $D \subsetneq \mathscr{C}_\Omega \subset \mathbb{R}_+^{N+1}$, consequently, $1 \le \sigma \le \frac{N+1}{N}$.
On the other hand, it is clear that the boundary of the extension cylinder $\mathscr{C}_\Omega$ possesses, by its very definition, the same regularity as the boundary of Ω, therefore, as we are considering Ω to be a smooth bounded domain, $\partial_L\mathscr{C}_\Omega$ satisfies the hypotheses above in Theorem 1.3.2. In fact, assuming that $\partial\Omega$ is a $\mathcal{C}^k$ manifold for some $k \ge 1$,

$$\lim_{\rho\to 0} \frac{|B_\rho(z)\backslash\Omega(z,\rho)|}{|B_\rho(z)|} = \frac{1}{2}, \tag{1.3.7}$$

for any $z \in \partial\Omega$. More generally, we can consider domains Ω such that $\partial\Omega$ is a Lipstchiz manifold. In this case (1.3.7) remains true replacing $\frac{1}{2}$ with certain constant $0 < c < 1$.
As an immediate consequence of Theorem 1.3.2 we obtain the following.

LEMMA 1.3.1. *Let $Z \in \Sigma_{\mathcal{D}}^*$ and $p \ge 2 - \epsilon_0$ for some $\epsilon_0 > 0$. Then, there exists $\overline{\rho} > 0$, such that for all $\rho < \overline{\rho}$ and any $U \in \mathcal{X}_{\Sigma_{\mathcal{D}}}^s(\mathscr{C}_\Omega)$ we have*

$$\|U\|_{L^{\sigma p}(\mathscr{C}_\Omega(Z,\rho),y^{1-2s}dxdy)} \le c_s\rho|B_\rho|_{y^{1-2s}}^{\frac{1}{p}\left(\frac{1}{\sigma}-1\right)}\|\nabla U\|_{L^p(\mathscr{C}_\Omega(Z,\rho),y^{1-2s}dxdy)}, \tag{1.3.8}$$

with $1 \le \sigma \le \frac{N+1}{N} + \delta$ for some $\delta > 0$ and c_s depending on N, p and the weight y^{1-2s}.

Although Theorem 1.1.1 addresses smooth domains, based on [**80**], we will be able to prove most of the results in this section under more general hypotheses on $\partial\Omega$. Then, we relax the smoothness hypotheses on $\partial\Omega$ and establish inequality (1.3.8) for functions in $\mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ and, given some point $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{D},R}$, also for functions in $H^1(\mathscr{C}_\Omega(Z,\rho), y^{1-2s}dxdy)$ vanishing on suitable sets.

Definition 1.3.2. *Given $p \geq 2 - \epsilon_0$ and an open bounded set A, we define $\mathcal{F}(\beta_s, A)$ as the family of sets $B \subset \overline{A}$ such that, for all $U \in H^1(A, y^{1-2s}dxdy)$ vanishing on B,*

$$\|U\|_{L^{\sigma p}(A,y^{1-2s}dxdy)} \leq \beta_s \, diam(A)|A|^{\frac{1}{p}(\frac{1}{\sigma}-1)}_{y^{1-2s}} \|\nabla U\|_{L^p(A,y^{1-2s}dxdy)}, \tag{1.3.9}$$

for some $\beta_s > 0$ depending on N, p and the weight y^{1-2s}, and $1 \leq \sigma \leq \frac{N+1}{N} + \delta$ for some $\delta > 0$.

Definition 1.3.3. *Given $0 < \lambda < 1$, we say that $\mathscr{C}_\Omega$ is a λ-admissible set if there exist $\beta_s, \zeta > 0$ such that for all $Z \in \partial_L \mathscr{C}_\Omega$ exists $\overline{\rho}(Z) > 0$ such that for $0 < \rho < \overline{\rho}(Z)$ we have one of the following conditions:*

1. *For any $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$: $\{U = 0\} \cap \overline{\mathscr{C}_\Omega(Z,\rho)} \in \mathcal{F}(\beta_s, \mathscr{C}_\Omega(Z,\rho))$.*
2. *For any open set $E \subset \mathscr{C}_\Omega(Z,\rho)$ such that $|E|_{y^{1-2s}} > \lambda|\mathscr{C}_\Omega(Z,\rho)|_{y^{1-2s}}$ we have $E \in \mathcal{F}(\beta_s, \mathscr{C}_\Omega(Z,\rho))$. Moreover,*

$$|\mathscr{C}_\Omega(Z,\rho)| \geq \zeta |B_\rho(Z)|. \tag{1.3.10}$$

In addition, if there exists $\delta > 0$ such that $\overline{\rho}(Z) > \delta$ for all $Z \in \partial_L \mathscr{C}_\Omega$ we say that $\mathscr{C}_\Omega$ is an uniform λ-admissible set.

Examples of λ-admissible sets are smooth domains or, even, Lipschitz domains such that Γ is a smooth or Lipschitz $(N-2)$-dimensional manifold splitting $\partial\Omega$ into smooth or Lipschitz manifolds $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$. Furthermore, those types of sets are examples of uniform λ-admissible sets.

Let us note that points on $\partial_L \mathscr{C}_\Omega$ that verify (1) lie in the Dirichlet boundary part, while those related to (2) are in the Neumann boundary part. On the other hand, (1.3.10) is always satisfied for interior points and, therefore, the λ-admissibility condition focus only on boundary points. It is also clear that, if $B_1 \supset B_2$ and $B_2 \in \mathcal{F}(\beta_s, A)$, then $B_1 \in \mathcal{F}(\beta_s, A)$. Moreover, because of construction, the lateral boundary of the extension cylinder is determined by $\partial\Omega$, then, if $\mathscr{C}_\Omega$ is a λ-admissible set, the number $\overline{\rho}(Z)$ does not depend on the y variable.

Remark 1.3.1. *Given an open set $A \subset \mathbb{R}^N$ and some $\beta > 0$, consider the family*

$$\mathcal{S}(\beta, A) := \{B \subset \overline{A} : |u(x)| \leq \beta \int_A \frac{|\nabla u(y)|}{|x-y|^{N-1}} dy, \ \forall u \in C^1(\overline{A}) \text{ vanishing on } B\}.$$

*As it can be seen in the proof of Theorem 1.3.1 in [**48**], the family $\mathcal{S}(\beta, A)$ plays a mayor role*[2] *in order to establish inequality* (1.3.6)*. In fact, $\mathcal{S}(\beta, A) \subset \mathcal{F}(\beta_s, A)$ for some $\beta_s > 0$ depending on β, N and the weight y^{1-2s}. Moreover, we can write $\beta_s = \beta c_s$ for a constant $c_s > 0$ depending on N and the weight y^{1-2s}.*

[2]Together with some integrability properties of the weights $\omega \in A_p$. Note as well that we have described the family $\mathcal{S}(\beta, A)$ in terms of the ambient space $\mathbb{R}^N$ according to the statement of Theorem 1.3.1.

As long as the family $\mathcal{S}(\beta, A)$ is involved, let us recall the following, see [**80**, §4]. *Given $x_0 \in A$ and a closed set $E \subset A$, let us consider the cone $\mathcal{V}_{x_0}(E) \subset A$ consisting on all rays starting at x_0 and ending at some point $P \in E$. This cone intersects the unitary sphere $\mathbb{S}_{N-1}(x_0)$ at some set $\mathscr{S}_{x_0}$ of positive $(N-1)$-dimensional Lebesgue measure. We define*

$$\Pi(x_0, E, A) = |\mathcal{V}_{x_0}(E) \cap \mathbb{S}_{N-1}(x_0)| = |\mathscr{S}_{x_0}|.$$

THEOREM 1.3.3 ([**80**], Theorem 10.2). *Assume that there exists $\varphi > 0$ such that for all $x_0 \in A$, we have $\Pi(x_0, B, A) \geq \varphi$. Then, there exists a positive constant $\beta \leq 2/\varphi$, such that $B \in \mathcal{S}(\beta, A)$.*

As a consequence, we have the following.

(i) *Assume that there exists $0 < \zeta, \lambda_L < 1$ such that for all $z \in \Sigma_{\mathcal{N}}$ exists $\overline{\rho}(z) > 0$ such that for all $\rho < \overline{\rho}(z)$ we have $|\Omega(z, \rho)| \geq \zeta |B_\rho(z)|$. Moreover, assume that*

$$|\mathcal{V}_x(\partial\Omega(z, \rho))| > (1 - \lambda_L)|\Omega(z, \rho)|, \text{ for all } x \in \Omega(z, \rho).$$

Then, for any $\mu > \lambda_L$ and for all subsets $E \subset \Omega(z, \rho)$, $|E| > \mu|\Omega(z, \rho)|$, we have $E \in \mathcal{S}(\beta, \Omega(z, \rho))$ for $\beta = \frac{C(N)}{\zeta(\mu - \lambda_L)}$, (see [**80**, §4 Sec.10-11]).

(ii) *Assume that there exists $\varphi > 0$ and $\overline{\rho} > 0$, such that for all $0 < \rho < \overline{\rho}$ and $z \in \Sigma_{\mathcal{D}} \setminus \Gamma$ we have*

$$\Pi(x, \Sigma_{\mathcal{D}} \cap B_\rho(z), \Omega(z, \rho)) \geq \varphi, \ \forall x \in \Omega(z, \rho).$$

Then, $\Sigma_{\mathcal{D}} \cap \Omega(z, \rho) \in \mathcal{S}(\beta, \Omega(z, \rho))$ for $\beta \leq \frac{2}{\varphi}$, (see [**80**, §4 Sec.12]).

(iii) *Assume that Γ is a $(N-2)$-dimensional Lipschitz manifold such that, for all $z \in \Gamma$ and $0 < \rho < \overline{\rho}$, with $\overline{\rho} > 0$ given in (ii), there exists a bi-Lipschitz transform*[3] *T such that*

- $T\left[\Omega(z, \rho)\right] = \mathcal{O}_\rho \subset \{x \in \mathbb{R}^N : x_{N-1} > 0, x_N > 0\}$,
- $T\left[\Sigma_{\mathcal{D}} \cap B_\rho(z)\right] = \overline{\mathcal{O}}_\rho \cap \{x_N = 0\}$,
- $T\left[\Sigma_{\mathcal{N}} \cap B_\rho(z)\right] = \overline{\mathcal{O}}_\rho \cap \{x_{N-1} = 0\}$,
- $T\left[\Gamma\right] = \overline{\mathcal{O}}_\rho \cap \{x_N = 0, x_{N-1} = 0\}$.

Assume in addition that, for all $x \in \mathcal{O}_\rho$, $\Pi(x, T\left[\Sigma_{\mathcal{D}}\right], \mathcal{O}_\rho) \geq \varphi$ for some $\varphi > 0$. Then, $\Sigma_{\mathcal{D}} \cap B_\rho(y) \in \mathcal{S}(\beta, \Omega(y, \rho))$ for some[4] *$\beta \leq \frac{C}{\varphi}$ for a positive constant C depending on N and the Lipschitz constant of T.*

Thus, if we assume that $\mathscr{C}_\Omega$ satisfies[5] *the hypotheses above in (i)-(iii), we obtain the following:*

(i^*) *Under the hypotheses of assertion (i), given $Z \in \overline{\mathscr{C}}_\Omega \setminus \Sigma^*_{\mathcal{D}}$ and $0 < \rho < \overline{\rho}(Z)$, inequality* (1.3.9) *holds with $\beta_s \geq \frac{c_s}{\zeta(\mu - \lambda_L)}$, where c_s depends on N and the weight y^{1-2s}, for any $U \in H^1(\mathscr{C}_\Omega(Z, \rho), y^{1-2s}dxdy)$ such that $|\{U = 0\} \cap \mathscr{C}_\Omega(Z, \rho)| \geq \mu|\mathscr{C}_\Omega(Z, \rho)|$ with $\mu > \lambda_L$.*

[3] That is, a biyective transformation $x' = T(x)$ such that, for some $c > 0$ we have $c^{-1}|x' - x''| \leq |T(x') - T(x'')| \leq c|x' - x''|$.

[4] Since, $\min\{c^{-N}, c^N\} \leq |DT(x')| \leq \max\{c^{-N}, c^N\}$ it follows that if $B \in \mathcal{S}(\beta, A)$ then $T(B) \in \mathcal{S}(\beta', T(A))$ with $\min\{c^{-N}, c^N\}\beta \leq \beta' \leq \max\{c^{-N}, c^N\}\beta$.

[5] In fact, it is sufficient to assume that Ω satisfies the hypotheses in assertions (i)-(iii) because then, by its very construction, the extension cylinder satisfies the same hypotheses with different constants.

(ii^*) *Under the hypotheses of assertion (ii), given a point $Z \in \Sigma_{\mathcal{D}}^* \setminus \Gamma^*$, for all functions $U \in H^1_{\Sigma_{\mathcal{D}}^*}(\mathscr{C}_\Omega(Z,\rho), y^{1-2s}dxdy)$ and $0 < \rho < \overline{\rho}$, the set $\{U = 0\} \cap \mathscr{C}_\Omega(Z,\rho) \in \mathcal{F}(\beta_s, \mathscr{C}_\Omega(Z,\rho))$ with $\beta_s \leq \frac{c_s}{\varphi}$, where $\varphi > 0$ is defined as in (ii) and c_s depends on N and the weight y^{1-2s}.*

(iii^*) *Under the hypotheses of assertion (iii), it is satisfied that for all points $Z \in \Gamma^*$, $\Sigma_{\mathcal{D}}^* \cap \mathscr{C}_\Omega(Z,\rho) \in \mathcal{F}(\beta, \mathscr{C}_\Omega(Z,\rho))$ with $\beta_s \leq \frac{c_s}{\varphi}$, where $\varphi > 0$ defined as in (ii) and c_s depends on N and the weight y^{1-2s}.*

Now, let us consider a weight $\omega \in A_p$. It can be proved, see [**81**, Ch. V §1.7]*, that for any $0 < \alpha < 1$ there exists $0 < \alpha_\omega < 1$ such that for all balls B and all subsets $E \subset B$,*

$$|E| \geq \alpha |B| \Rightarrow |E|_\omega \geq \alpha_\omega |B|_\omega.$$

Moreover, $\alpha_\omega = \frac{\alpha^p}{b_\omega}$ and $b_\omega \geq 1$ is a constant, depending on N and the weight ω, known as the A_p-constant. Then, taking $\zeta_s = \frac{\zeta^2}{b_s} < 1$, from (1.3.10) *we obtain*

$$|\mathscr{C}_\Omega(Z,\rho)|_{y^{1-2s}} \geq \zeta_s |B_\rho(Z)|_{y^{1-2s}}. \tag{1.3.11}$$

Finally, if we set $\mu > \lambda_L$ as in (i^*), we obtain

$$|E| \geq \mu |\mathscr{C}_\Omega(Z,\rho)| \geq \mu\zeta |B_\rho(Z)| \Rightarrow |E|_{y1-2s} \geq \lambda |\mathscr{C}_\Omega(Z,\rho)|_{y1-2s},$$

with $\lambda = \mu^2 \zeta_s$. And we conclude that $\mathscr{C}_\Omega$ is a λ-admissible set for all $\lambda_L^2 \zeta_s < \lambda < 1$. As a consequence, if we assume that $\Sigma_{\mathcal{N}}$ is a Lipschitz manifold with small Lipschitz constant, and $\Sigma_{\mathcal{D}}$, Γ are Lipschitz manifolds, then $\mathscr{C}_\Omega$ is an uniform λ-admissible set for all $\lambda_L^2 \zeta_s < \lambda < 1$. Hence, if Ω is a smooth bounded domain and $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ satisfy the hypotheses ($\mathfrak{B}$)*, then $\mathscr{C}_\Omega$ is an uniform λ-admissible set for all $0 < \lambda < 1$.*

With this scheme in mind, we focus first on find a bound for solutions to (1.3.1) in terms of the measure of the domain $\Omega(z,R)$, the datum f and a positive constant $C = C(N, s, |\Sigma_{\mathcal{D}}|)$. This is done adapting to our framework [**62**, Theorem B.2]. Next, we establish bounds on the oscillation of functions $w(x)$ satisfying (1.3.2). This is done using arguments similar to those of [**80**, Theorem 8.5] and [**62**, Theorem D.5]. To accomplish this step we work with the extended problem (1.3.4). Gathering together these results, we will be able to prove the local Hölder regularity of solutions to problem (P^s).

THEOREM 1.3.4. *Let u be a solution of (P^s) with $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. Then, there exists a positive constant $C = C(N, s, |\Sigma_{\mathcal{D}}|)$ such that*

$$\max_{x \in \Omega} u(x) \leq C \|f\|_{L^p(\Omega)} |\Omega|^{\frac{2s}{N} - \frac{1}{p}}.$$

In the proof of Theorem 1.3.4 we make use of the following technical result [**62**, Lemma B.1].

LEMMA 1.3.2. *Let $\varphi(k)$ be a nonnegative and nonincreasing function defined for $k \geq k_0$ such that*

$$\varphi(h) \leq [C/(h-k)^a] |\varphi(k)|^b, \qquad k < h,$$

where C, a, b are positive constants with $b > 1$. Then, $\varphi(k_0 + d) = 0$, with

$$d^a = 2^{\frac{ab}{b-1}} C |\varphi(k_0)|^{b-1}.$$

PROOF OF THEOREM 1.3.4. Let us take $k \geq 0$, $U = E_s[u]$ and $\psi = (U-k)_+ \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ as a test function in (1.2.4). Using the trace inequality (1.2.9) together with the Hölder inequality, we get,

$$\begin{aligned}\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \nabla U \nabla \psi dx dy = \kappa_s \int_{A_+^*(k)} y^{1-2s} |\nabla U|^2 dx dy &= \int_{A_+(k)} (U(x,0)-k) f(x) dx \\ &\leq \left(\int_{A_+(k)} |f|^2 dx \right)^{\frac{1}{2}} \left(\int_{A_+(k)} |U(x,0)-k|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{A_+(k)} |f|^2 dx \right)^{\frac{1}{2}} \left(C_{\mathcal{D}}^{-1} |A_+(k)|^{\frac{2s}{N}} \int_{A_+^*(k)} y^{1-2s} |\nabla U|^2 dx dy \right)^{\frac{1}{2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{A_+^*(k)} y^{1-2s} |\nabla U|^2 dx dy &\leq C_{\mathcal{D}}^{-1} \kappa_s^{-2} |A_+(k)|^{\frac{2s}{N}} \int_{A_+(k)} |f|^2 dx \\ &\leq C_{\mathcal{D}}^{-1} \kappa_s^{-2} \|f\|^2_{L^p(\Omega)} |A_+(k)|^{1-\frac{2}{p}+\frac{2s}{N}}.\end{aligned} \tag{1.3.12}$$

Applying now the trace inequality (1.2.9) to the left-hand side of (1.3.12) and noticing that for $h > k$,

$$(h-k)^2 |A_+(h)|^{\frac{2}{2_s^*}} \leq \left(\int_{A_+(k)} |U(x,0)-k|^{2_s^*} \, dx \right)^{\frac{2}{2_s^*}},$$

we conclude

$$(h-k)^2 |A_+(h)|^{\frac{2}{2_s^*}} \leq (C_{\mathcal{D}} \kappa_s)^{-2} \|f\|^2_{L^p(\Omega)} |A_+(k)|^{1-\frac{2}{p}+\frac{2s}{N}},$$

Taking $\varphi(h) = |A_+(h)|$, it follows that

$$\varphi(h) \leq \frac{(C_{\mathcal{D}} \kappa_s)^{-2}}{(h-k)^{2_s^*}} \|f\|^{2_s^*}_{L^p(\Omega)} \left[\varphi(k)\right]^{\left(1-\frac{2}{p}+\frac{2s}{N}\right)\frac{2_s^*}{2}}.$$

Applying now Lemma 1.3.2 with $a = 2_s^*$ and $b = \left(1 - \frac{2}{p} + \frac{2s}{N}\right)\frac{2_s^*}{2} > 1$, we find $|\varphi(k_0+d)| = 0$ with $d = C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)} |\varphi(k_0)|^{\frac{b-1}{a}}$, and $\frac{b-1}{a} = \frac{2s}{N} - \frac{1}{p}$, i.e.

$$U(x,0) \leq k_0 + C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)} |A_+(k_0)|^{\frac{2s}{N}-\frac{1}{p}}, \quad \text{a.e. in } \Omega,$$

for any $k_0 \geq 0$, and we conclude

$$u(x) \leq C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)} |\Omega|^{\frac{2s}{N}-\frac{1}{p}}, \quad \text{a.e. in } \Omega.$$

□

Let v be the solution of (1.3.1) and $V = E_s[v]$ the solution of (1.3.3). Since the function $(V-k)_+ \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ for any $k \geq 0$, repeating the steps above in Theorem 1.3.4, we find

$$\max_{x \in \Omega(z,R)} v(x) \leq C(N,s,|\Sigma_{\mathcal{D}}|)\|f\|_{L^p(\Omega)} |\Omega(z,R)|^{\frac{2s}{N}-\frac{1}{p}}. \tag{1.3.13}$$

Now we turn our attention to the study of the behavior of solutions to the homogeneous problem (1.3.4).

LEMMA 1.3.3 (Caccioppoli inequality). *Assume that $z \in \overline{\Omega}$ and $R > 0$ and suppose that the function $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ is a solution of problem (1.3.4). Then, for any $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ and $0 < \rho < r < \widetilde{\rho}(Z)$, we have*

$$\int_{\mathscr{C}_{\Omega(z,R)}(Z,\rho)} y^{1-2s}|\nabla W|^2 dxdy \leq \frac{C}{(r-\rho)^2}\int_{\mathscr{C}_{\Omega(z,R)}(Z,r)} y^{1-2s} W^2 dxdy,$$

for some $C > 0$.

PROOF. Using $\psi = \eta^2 W$ as a test function in (1.3.5), with $\eta \in C^1(\mathscr{C}_{\Omega(z,R)})$ and vanishing on $\partial_L \mathscr{C}_{\Omega(z,R)} \backslash (\Sigma^*_{\mathcal{D},R} \cup \Sigma^*_{\mathcal{N},R})$ so that ψ vanish on $\partial_L \mathscr{C}_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{N},R}$, we find

$$\begin{aligned}\int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}\eta^2|\nabla W|^2 dxdy &= -2\int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}\langle \eta\nabla W, W\nabla\eta\rangle dxdy \\ &\leq 2\left(\frac{1}{2\varepsilon}\int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}|\nabla\eta|^2 W^2 dxdy + \frac{\varepsilon}{2}\int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}\eta^2|\nabla W|^2 dxdy\right),\end{aligned}$$

for some $0 < \varepsilon < 1$. To complete the proof, given $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ and $\rho < r < \widetilde{\rho}(Z)$ it is enough to set η such that

$$\eta \equiv 1 \text{ in } B_\rho(Z), \qquad \eta \equiv 0 \text{ in } B^c_r(Z) \qquad \text{and} \qquad |\nabla\eta| \leq \frac{c}{(r-\rho)}.$$

$\square$

LEMMA 1.3.4. *Let $p \geq 2 - \epsilon_0$ for some $\epsilon_0 > 0$ and $U \in \mathcal{X}^s(\mathscr{C}_\Omega)$ such that $\{U = 0\} \in \mathcal{F}(\beta, A)$ for $A \subset \overline{\mathscr{C}}_\Omega$. Then,*

$$\int_A y^{1-2s} U^p dxdy \leq \beta^p_s [diam(A)]^p |A|^{(\frac{1}{\sigma}-1)}_{y^{1-2s}} \int_A y^{1-2s}|\nabla U|^p dxdy \cdot |\{(x,y) \in A : U \neq 0\}|^{\frac{1}{\sigma'}}_{y^{1-2s}},$$

with $\beta_s > 0$ depending on N, p and the weight y^{1-2s}, $1 \leq \sigma \leq \frac{N+1}{N} + \delta$ for some $\delta > 0$ and $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$.

PROOF. Using the Hölder inequality together with (1.3.9), we find

$$\begin{aligned}\int_A y^{1-2s} U^p dxdy &\leq \left(\int_A y^{1-2s} U^{\sigma p} dxdy\right)^{\frac{1}{\sigma}} |\{(x,y) \in A : U \neq 0\}|^{\frac{1}{\sigma'}}_{y^{1-2s}} \\ &\leq \beta^p_s [\mathrm{diam}(A)]^p |A|^{(\frac{1}{\sigma}-1)}_{y^{1-2s}} \int_A y^{1-2s}|\nabla U|^p dxdy \cdot |\{(x,y) \in A : U \neq 0\}|^{\frac{1}{\sigma'}}_{y^{1-2s}}\end{aligned}$$

$\square$

As a consequence of Lemma 1.3.4 we prove the following results.

LEMMA 1.3.5. *Given $z \in \overline{\Omega}$ and $R > 0$, assume that $\mathscr{C}_{\Omega(z,R)}$ is a λ-admissible set and let $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$. Then, for any $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ and $0 < r < \overline{\rho}(Z)$, we have,*

$$\int_{A^*_+(k,r)} y^{1-2s}|U-k|^2 dxdy \leq \beta^2_s r^2 |B_r(Z)|^{(\frac{1}{\sigma}-1)}_{y^{1-2s}} \int_{A^*_+(k,r)} y^{1-2s}|\nabla U|^2 dxdy \cdot |A^*_+(k,r)|^{\frac{1}{\sigma'}}_{y^{1-2s}},$$

with $\beta_s > 0$ depending on N and the weight y^{1-2s} , $1 \leq \sigma \leq \frac{N+1}{N} + \delta$ for some $\delta > 0$ and $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, provided that

(i) $k \in \mathcal{K}^+(Z)$ *if* $Z \in \Sigma^*_{\mathcal{D},R}$.
(ii) $k \in \mathcal{K}^+(Z)$ *is such that*

$$|A^*_+(k,r)|_{y^{1-2s}} \leq (1-\lambda)|\mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}},$$

if $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{D},R}$.

Proof. The proof follows as in [**80**, Theorem 6.1]. Given $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$, let us consider the function $t^+_k(U) = (U-k)_+$ that belongs to $\mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ for any $k \in \mathbb{R}$. Moreover, if $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ then $t^+_k(U) \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ for any $k \geq 0$. Thus, as $\mathscr{C}_{\Omega(z,R)}$ is a λ-admissible set,

(i) If $Z \in \Sigma^*_{\mathcal{D},R}$ then $\{t^+_k(U)=0\} \supseteq \{U=0\}$, so the set $\{t^+_k(U)=0\} \cap \mathscr{C}_{\Omega(z,R)}(Z,r)$ belongs to $\mathcal{F}(\beta_s, \mathscr{C}_{\Omega(z,R)}(Z,r))$.
(ii) If $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{D},R}$, then $\{t^+_k(U)=0\} \cap \mathscr{C}_{\Omega(z,R)}(Z,r) \equiv \mathscr{C}_{\Omega(z,R)}(Z,r) \backslash A^*_+(k,r)$ and, thus, $|\{t^+_k(U)=0\} \cap \mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}} > \lambda |\mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}}$. As a consequence, $\{t^+_k(U)=0\} \cap \mathscr{C}_{\Omega(z,R)}(Z,r) \in \mathcal{F}(\beta_s, \mathscr{C}_{\Omega(z,R)}(Z,r))$.

Then, using Lemma 1.3.4 with $p=2$, we conclude

$$\int_{A^*_+(k,r)} y^{1-2s}|U-k|^2 dxdy \leq \beta_s^2 r^2 |B_r(Z)|^{\left(\frac{1}{\sigma}-1\right)}_{y^{1-2s}} \int_{A^*_+(k,r)} y^{1-2s}|\nabla t^+_k(U)|^2 dxdy \cdot |A^*_+(k,r)|^{\frac{1}{\sigma'}}_{y^{1-2s}}.$$

□

Lemma 1.3.6. *Given* $z \in \overline{\Omega}$ *and* $R>0$, *assume that* $\mathscr{C}_{\Omega(z,R)}$ *is a* λ*-admissible set and let* $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$. *Then, for any* $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ *and* $0<r<\overline{\rho}(Z)$, *we have*

$$(h-k)^2 |A^*_+(h,r)|^{\frac{2}{q}}_{y^{1-2s}} \leq \beta_s^2 r^2 |B_r(Z)|^{2\left(\frac{1}{q}-\frac{1}{p}\right)}_{y^{1-2s}} |A^*_+(k,r) - A^*_+(h,r)|^{\frac{2}{p}-1}_{y^{1-2s}} \int_{A^*_+(k,r)} y^{1-2s}|\nabla U|^2 dxdy,$$

with $h>k$, $q = \frac{N+1}{N}(2-\epsilon)$ $p = 2-\epsilon_0$ *and* $\beta_s > 0$ *depending on* N, p *and the weight* y^{1-2s}; *provided that*

(i) $k \in \mathcal{K}^+(Z)$ *if* $Z \in \Sigma^*_{\mathcal{D},R}$.
(ii) $k \in \mathcal{K}^+(Z)$ *is such that*

$$|A^*_+(k,r)|_{y^{1-2s}} \leq (1-\lambda)|\mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}},$$

if $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{D},R}$.

Proof. Given $U \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ and $h>k$, let $t^+_{h,k}(U) = \{U\}^h - \{U\}^k$. Note that $t^+_{h,k}(U) \in \mathcal{X}^s(\mathscr{C}_{\Omega(z,R)})$ for any $k \in \mathbb{R}$. Moreover, if $U \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ then $t^+_{h,k}(U) \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ for any $h>k\geq 0$.

(i) If $Z \in \Sigma^*_{\mathcal{D},R}$, then $\{t^+_{h,k}(U)=0\} = \{t^+_k(U)=0\} \supset \{U=0\}$ and, therefore, $\{t^+_{h,k}(U)=0\} \cap \mathscr{C}_{\Omega(z,R)}(Z,r) \in \mathcal{F}(\beta_s, \mathscr{C}_{\Omega(z,R)}(Z,r))$.
(ii) If $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{D},R}$, then $\{t^+_{h,k}(U)=0\} \supset \{t^+_k(U)=0\}$. Repeating the arguments for (ii) in Lemma 1.3.5 we conclude $\{t^+_{h,k}(U)=0\} \cap \mathscr{C}_{\Omega(z,R)}(Z,r) \in \mathcal{F}(\beta_s, \mathscr{C}_{\Omega(z,R)}(Z,r))$.

Thus, using Lemma 1.3.4 with $\sigma = \frac{N+1}{N}$ and $p = 2 - \epsilon_0$ so that taking $q = \sigma p = \frac{N+1}{N}(2 - \epsilon_0)$ we obtain,

$$\left(\int_{\mathscr{C}_{\Omega(z,R)}(Z,r)} y^{1-2s} [t^+_{h,k}(U)]^q dxdy \right)^{\frac{1}{q}} \le \beta_s r |B_r(Z)|_{y^{1-2s}}^{\frac{1}{q}-\frac{1}{p}} \left(\int_{A^*_+(k,r) - A^*_+(h,r)} y^{1-2s} |\nabla U|^p dxdy. \right)^{\frac{1}{p}}.$$

At one hand, it is immediate that

$$(h-k)^2 |A^*_+(h,r)|_{y^{1-2s}}^{\frac{2}{q}} \le \left(\int_{\mathscr{C}_{\Omega(z,R)}(Z,r)} y^{1-2s} \left(t^+_{h,k}(U) \right)^q dxdy \right)^{\frac{2}{q}}.$$

On the other hand, thanks to Hölder inequality

$$\left(\int_{A^*_+(k,r) - A^*_+(h,r)} y^{1-2s} |\nabla U|^p dxdy \right)^{\frac{2}{p}} \le \int_{A^*_+(k,r)} y^{1-2s} |\nabla U|^2 dxdy \times |A^*_+(k,r) - A^*_+(h,r)|_{y^{1-2s}}^{\frac{2}{p}-1}.$$

□

Following [**80**, Theorem 8.1], we show the next result.

THEOREM 1.3.5. *Given $z \in \overline{\Omega}$ and $R > 0$, assume that $\mathscr{C}_{\Omega(z,R)}$ is a λ-admissible set and $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z,R)})$ is a solution of the homogeneous problem (1.3.4). Then, for any $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$, $0 < l < 1$ and $0 < r < \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$, there exists a positive constant $\Lambda = \Lambda(l)$ such that*

$$|A^*_+(k + ld, r - lr)| = 0,$$

where

$$d^2 \ge \frac{1}{\Lambda |B_r(Z)|_{y^{1-2s}}} \int_{A^*_+(k,r)} y^{1-2s} |W - k|^2 dxdy,$$

provided that $k \in \mathcal{K}^+(Z)$ is such that

$$|A^*_+(k,r)|_{y^{1-2s}} \le \Lambda |\mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}}. \tag{1.3.14}$$

In the proof of Theorem 1.3.5 we will use the next technical result [**62**, Lemma C.7].

LEMMA 1.3.7. *Assume that $\varphi(k,\rho)$ is a nonnegative function defined for $k \ge k_0$ and $0 < \rho \le r_0$ which is nonincreasing with respect to k, nondecreasing with respect to ρ and such that*

$$\varphi(h,\rho) \le \frac{C}{(h-k)^\alpha (r-\rho)^\gamma} [\varphi(k,r)]^\mu, \qquad k < h, \ \rho < r \le r_0,$$

where C, α, β, γ are positive constants with $\mu > 1$. Then, $\varphi(k_0 + ld, r_0 - lr_0) = 0$, with $0 < l < 1$ and

$$d^\alpha = \frac{2^{(\alpha+\gamma)\mu/(\mu-1)} C [\varphi(k_0, r_0)]^{\mu-1}}{l^{\alpha+\gamma} r_0^\gamma}.$$

PROOF OF THEOREM 1.3.5. Given $Z \in \overline{\mathscr{C}}_{\Omega(z,R)}$, $k_0 \in \mathcal{K}^+(Z)$ satisfying (1.3.14) and $k \ge k_0$, let us define

$$i(k,\rho) = \int_{A^*(k,\rho)} y^{1-2s} |W - k|^2 dxdy \quad \text{and} \qquad a(k,\rho) = |A^*_+(k,\rho)|_{y^{1-2s}}.$$

Observe that for $h > k$ we have

$$(h-k)^2|A_+^*(h,\rho)|_{y^{1-2s}} \le \int_{A_+^*(k,r)} y^{1-2s}|W-k|^2 dxdy. \tag{1.3.15}$$

Assume that $Z \in \Sigma_{\mathcal{D},R}^*$ and let $0 < r_0 < \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$. Then, due to Lemma 1.3.5 and Lemma 1.3.3, for all $r_0 - lr_0 \le \rho < r \le r_0$ and $h > k$, we have

$$\begin{aligned}
\int_{A_+^*(h,\rho)} y^{1-2s}|W-h|^2 dxdy &\le K_{\mathscr{C}_\Omega(\rho)} \left(\int_{A_+^*(h,\rho)} y^{1-2s}|\nabla W|^2 dxdy\right) |A_+^*(h,\rho)|_{y^{1-2s}}^{\frac{1}{\sigma'}} \\
&\le K_{\mathscr{C}_\Omega(\rho)} \left(\int_{A_+^*(k,\rho)} y^{1-2s}|\nabla W|^2 dxdy\right) |A_+^*(k,\rho)|_{y^{1-2s}}^{\frac{1}{\sigma'}} \\
&\le K_{\mathscr{C}_\Omega(\rho)} \left(\frac{1}{(r-\rho)^2}\int_{A_+^*(k,r)} y^{1-2s}|W-k|^2 dxdy\right) |A_+^*(k,r)|_{y^{1-2s}}^{\frac{1}{\sigma'}},
\end{aligned} \tag{1.3.16}$$

where $K_{\mathscr{C}_\Omega(r)} = \beta_s^2 r^2 |B_r(Z)|_{y^{1-2s}}^{\frac{1}{\sigma}-1}$, with $\beta_s > 0$ depending on N, the weight y^{1-2s} and the geometry of $\partial\Omega$; $1 \le \sigma \le \frac{N+1}{N} + \delta$ for some $\delta > 0$.
Assume now that $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma_{\mathcal{D},R}^*$. Taking in mind (1.3.11), let $\Lambda = \Lambda(l) > 0$ satisfying

$$\frac{\Lambda}{\zeta_s(1-l)^{N+2(1-s)}} \le (1-\lambda).$$

Therefore, given $h \ge k_0$ and $r_0 - lr_0 \le \rho \le r_0$, we find

$$\begin{aligned}
|A_+^*(h,\rho)|_{y^{1-2s}} &\le |A_+^*(k_0,r_0)|_{y^{1-2s}} \le \Lambda|\mathscr{C}_{\Omega(z,R)}(Z,r_0)|_{y^{1-2s}} \le \Lambda|B_{r_0}(Z)|_{y^{1-2s}} \\
&\le \frac{\Lambda}{(1-l)^{N+2(1-s)}}|B_\rho(Z)|_{y^{1-2s}} \le \frac{\Lambda}{\zeta_s(1-l)^{N+2(1-s)}}|\mathscr{C}_{\Omega(z,R)}(Z,\rho)|_{y^{1-2s}} \\
&\le (1-\lambda)|\mathscr{C}_{\Omega(z,R)}(Z,\rho)|_{y^{1-2s}}.
\end{aligned}$$

Using Lemma 1.3.5 and Lemma 1.3.3 we conclude (1.3.16). As a consequence, for any $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$,

$$i(h,\rho) \le \frac{K_{\mathscr{C}_\Omega(\rho)}}{(r-\rho)^2} i(k,r)[a(k,r)]^{\frac{1}{\sigma'}}, \tag{1.3.17}$$

with $r_0 - lr_o \le \rho < r \le r_0$, and $h > k \ge k_0$ with $k_0 \in \mathcal{K}^+(Z)$ satisfying (1.3.14). Moreover, because of $|B_{\mu r}(Z)|_{y^{1-2s}} = \mu^{N+2(1-s)}|B_r(Z)|_{y^{1-2s}}$, setting the constant $\varsigma = 2 + \left(\frac{1}{\sigma}-1\right)(N+2(1-s))$, we find $K_{\mathscr{C}_\Omega(\mu r)} = \mu^\varsigma K_{\mathscr{C}_\Omega(r_0)}$. If we let $1 < \sigma \le 1 + \frac{2}{N-2s}$, so that $\varsigma > 0$, then $K_{\mathscr{C}_\Omega(r)} \le K_{\mathscr{C}_\Omega(r_0)}$ for all $r < r_0$. Hence, from (1.3.17), we obtain

$$i(h,\rho) \le \frac{K_{\mathscr{C}_\Omega(r_0)}}{(r-\rho)^2} i(k,r)[a(k,r)]^{\frac{1}{\sigma'}}, \qquad \rho < r \le r_0,\ h > k \ge k_0, \tag{1.3.18}$$

and $K_{\mathscr{C}_\Omega(r_0)} = \beta_s^2 r_0^2 |B_{r_0}(Z)|_{y^{1-2s}}^{\frac{1}{\sigma}-1}$ fixed. Set $\xi + 1 = \theta\xi$ and $\frac{\xi}{\sigma'} = \theta$, so that θ is the positive solution of the equation $\theta^2 - \theta - \frac{1}{\sigma'} = 0$, i.e. $\theta = (1/2) + \sqrt{\frac{1}{4} + \frac{1}{\sigma'}} > 1$. Assume in addition that the constant Λ satisfies

$$\Lambda^{\frac{\theta}{2}} \leq \frac{l^{(\xi+1)}}{\beta_s^{\xi} 2^{(\xi+1)\frac{\theta}{\theta-1}}}. \tag{1.3.19}$$

From (1.3.15) and (1.3.18), we obtain

$$|i(h,\rho)|^{\xi}|a(h,\rho)| \leq \frac{K^{\xi}_{\mathscr{C}_{\Omega(r_0)}}}{(r-\rho)^{2\xi}(h-k)^2}|i(k,r)|^{\xi+1}|a(k,r)|^{\frac{\xi}{\sigma'}}.$$

Then, taking $\varphi(k,\rho) = |i(k,\rho)|^{\xi}|a(k,\rho)|$, it follows that

$$\varphi(h,\rho) \leq \frac{K^{\xi}_{\mathscr{C}_{\Omega(r_0)}}}{(r-\rho)^{2\xi}(h-k)^2}[\varphi(k,r)]^{\theta}, \quad h > k \geq k_0,\ \rho < r \leq r_0.$$

Using Lemma 1.3.7 with $\alpha = 2$, $\mu = \theta$, $\gamma = 2\xi$, we conclude that

$$\varphi(k_0 + ld_0, r_0 - lr_0) = 0,$$

for any $k_0 \in \mathcal{K}^+(Z)$ satisfying (1.3.14), $0 < r_0 < \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$ and $0 < l < 1$, with

$$\begin{aligned}
d_0 &= \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}}{l^{\xi+1}} \frac{K^{\xi/2}_{\mathscr{C}_{\Omega(r_0)}}[\varphi(k_0,r_0)]^{\frac{\theta-1}{2}}}{r_0^{\xi}} = \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}}{l^{\xi+1}} \left(\beta_s^2 r_0^2 |B_{r_0}(Z)|^{\frac{1}{\sigma}-1}_{y^{1-2s}}\right)^{\xi/2} \frac{[\varphi(k_0,r_0)]^{\frac{\theta-1}{2}}}{r_0^{\xi}} \\
&= \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}\beta_s^{\xi}}{l^{\xi+1}} \frac{[\varphi(k_0,r_0)]^{\frac{\theta-1}{2}}}{|B_{r_0}(Z)|^{\frac{\xi}{2\sigma'}}_{y^{1-2s}}} = \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}\beta_s^{\xi}}{l^{\xi+1}} \frac{[\varphi(k_0,r_0)]^{\frac{\theta-1}{2}}}{|B_{r_0}(Z)|^{\frac{\theta}{2}}_{y^{1-2s}}} \\
&= \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}\beta_s^{\xi}}{l^{\xi+1}|B_{r_0}(Z)|^{\frac{1}{2}}_{y^{1-2s}}} \left|\frac{\varphi(k_0,r_0)}{|B_{r_0}(Z)|_{y^{1-2s}}}\right|^{\frac{\theta-1}{2}} \\
&= \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}}{l^{\xi+1}}\beta_s^{\xi} \left(\frac{1}{|B_{r_0}(Z)|_{y^{1-2s}}} \int_{A_+^*(k,r_0)} y^{1-2s}|W-k_0|^2 dxdy\right)^{\frac{1}{2}} \left(\frac{|A_+^*(k_0,r_0)|_{y^{1-2s}}}{|B_{r_0}(Z)|_{y^{1-2s}}}\right)^{\frac{\theta-1}{2}} \\
&= \frac{2^{\frac{(\xi+1)\theta}{\theta-1}}}{l^{\xi+1}}\beta_s^{\xi}\Lambda^{\frac{\theta}{2}} \left(\frac{1}{\Lambda|B_{r_0}(Z)|_{y^{1-2s}}} \int_{A_+^*(k,r_0)} y^{1-2s}|W-k_0|^2 dxdy\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{\Lambda|B_{r_0}(Z)|_{y^{1-2s}}} \int_{A_+^*(k,r_0)} y^{1-2s}|W-k_0|^2 dxdy\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $|A_+^*(k_0+ld, r_0-lr_0)|_{y^{1-2s}} = 0$ implies $|A_+^*(k_0+ld, r_0-lr_0)| = 0$ the proof is complete. $\square$

Using the function $t_k^-(U) = U - \{U\}_k = (U-k)_-$ and repeating the arguments above in Lemma 1.3.3, Lemma 1.3.5 and Theorem 1.3.5 we can establish the following.

THEOREM 1.3.6. *Given $z \in \overline{\Omega}$ and $R > 0$, assume that $\mathscr{C}_{\Omega(z,R)}$ is a λ-admissible set and let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z,R)})$ be a solution of the homogeneous problem (1.3.4). Then, for any $Z \in \mathscr{C}^{\circ}_{\Omega(z,R)}$, $0 < l < 1$ and $0 < r < \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$, there exist a positive constant $\Lambda = \Lambda(l)$ such that*

$$|A_-^*(k - ld, r - lr)| = 0,$$

where

$$d^2 \geq \frac{1}{\Lambda|B_r(Z)|_{y^{1-2s}}} \int_{A_+^*(k,r)} y^{1-2s}|W-k|^2 dxdy,$$

provided that $k \in \mathcal{K}^-(Z)$ *is such that*

$$|A_-^*(k,r)|_{y^{1-2s}} \leq \Lambda |\mathscr{C}_{\Omega(z,R)}(Z,r)|_{y^{1-2s}}.$$

As a consequence of the two former results, we obtain an L^∞ bound on solutions to problem (1.3.2).

Corollary 1.3.1. *Given* $z \in \overline{\Omega}$ *and* $R > 0$, *assume that* $\mathscr{C}_{\Omega(z,R)}$ *is a* λ*-admissible set and* $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ *is a solution of the homogeneous problem* (1.3.4) *and consider the set* $\mathscr{C}^m_{\Omega(z,R/2)} = \mathscr{C}_{\Omega(z,R/2)} \cap \{y < m\}$ *with* $m > 0$. *Then,* $W \in L^\infty(\overline{\mathscr{C}^m_{\Omega(z,R/2)}})$ *for any finite* $m > 0$. *In particular, if* $w \in H^s_{\Sigma_{\mathcal{D},R}}(\Omega(z,R))$ *is the solution of problem* (1.3.2), *we conclude that* $w \in L^\infty(\overline{\Omega(z,R/2)})$.

Proof. First, let us prove that $w \in L^\infty(\overline{\Omega(z,R/2)})$ with w satisfying problem (1.3.2). Let $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ a solution of problem (1.3.4) so that $w = W(x,0)$ satisfies (1.3.2). Since $\overline{\Omega(z,R/2)}$ is a closed bounded set, there exists $Z_i = (z_i, 0) \in \mathscr{C}^\circ_{\Omega(z,R)}$, $i = 1, 2, \ldots, M$; such that

$$\overline{\Omega(z,R/2)} = \left(\bigcup_{i=1}^{M} \mathscr{C}^\circ_{\Omega(z,R)}(Z_i, r_i/2) \right) \cap \{y = 0\}, \tag{1.3.20}$$

with $0 < r_i < \{\widetilde{\rho}(Z_i), \overline{\rho}(Z_i)\}$. Let $\overline{k} > 0$ and $\hat{k} < 0$ such that,

$$|A_+^*(\overline{k}, r_i)| \leq \Lambda |\mathscr{C}_{\Omega(z,R)}(Z_i, r_i)|,$$
$$|A_-^*(\hat{k}, r_i)| \leq \Lambda |\mathscr{C}_{\Omega(z,R)}(Z_i, r_i)|,$$

for all $i = 1, 2, \ldots, M$. Then, applying Theorem 1.3.5 and Theorem 1.3.6 we conclude that, given $X \in \mathscr{C}_{\Omega(z,R)}(Z_i, r_i)$ for some $i = 1, 2, \ldots, M$; we have

$$\kappa_m := \hat{k} - ld \leq W(X) \leq \kappa_M := \overline{k} + ld, \tag{1.3.21}$$

with

$$d^2 \geq \frac{1}{\Lambda|B_r(Z)|_{y^{1-2s}}} \int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s}|W|^2 dxdy,$$

for any $0 < r < \min_{i=1,\ldots,M} r_i$. In particular, because of (1.3.20), the former inequality holds for all points $X = (x,0)$ with $x \in \overline{\Omega(z,R/2)}$ and we are done.
As $\mathscr{C}_{\Omega(z,R/2)}$ is an unbounded domain, if we repeat the steps above in order to prove that $W \in L^\infty(\overline{\mathscr{C}}_{\Omega(z,R/2)})$ from (1.3.21), the numbers $\hat{k}, \overline{k}$ may diverge when considering a covering sequence $\{Z_i\}_{i\in\mathbb{N}}$. Nevertheless, it is clear that given any finite truncation of the extension cylinder, $\mathscr{C}^m_{\Omega(z,R/2)} = \mathscr{C}_{\Omega(z,R/2)} \cap \{y < m\}$, there exists a finite covering sequence and hence, we conclude $W \in L^\infty(\overline{\mathscr{C}^m_{\Omega(z,R/2)}})$ for all finite $m > 0$. $\square$

REMARK 1.3.2. *Thanks to* (1.2.2) *and* (1.3.9) *with* $p = 2$ *and* $\sigma = 1$, *we can bound the constant* $d > 0$ *in Corollary 1.3.1 as follows,*

$$\begin{aligned} d^2 =& \frac{\beta_s^2 r^2}{\kappa_s \Lambda |B_r|_{y^{1-2s}}} \|w\|^2_{H^s_{\Sigma_{\mathcal{D}}(\Omega)}} = \frac{\beta_s^2 r^2}{\Lambda |B_r|_{y^{1-2s}}} \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla W|^2 dxdy \\ &\geq \frac{1}{\Lambda |B_r(Z)|_{y^{1-2s}}} \int_{\mathscr{C}_{\Omega(z,R)}} y^{1-2s} |W|^2 dxdy. \end{aligned}$$

We focus now on the oscillation of solutions $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ to problem (1.3.4). Let us set

$$m(\rho) = \inf_{X \in \overline{\mathscr{C}}_{\Omega(z,R)}(Z,\rho)} W(X) \qquad \text{and} \qquad M(\rho) = \sup_{X \in \overline{\mathscr{C}}_{\Omega(z,R)}(Z,\rho)} W(X).$$

and define the oscillation function as

$$\omega(\rho) := M(\rho) - m(\rho).$$

THEOREM 1.3.7. *Given* $z \in \overline{\Omega}$ *and* $R > 0$, *let* $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ *and assume that* $\mathscr{C}_{\Omega(z,R)}$ *is a* λ*-admissible set with* $\lambda \leq 1/2$ *and* $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ *is a solution of the homogeneous problem* (1.3.4). *Moreover, given* $0 < 4\rho < \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$ *let* $0 < \eta < 1$ *such that,*

(i) $(M(4\rho) - \eta\omega(4\rho), +\infty) \subset \mathcal{K}^+(Z)$,
(ii) $|A^*_+(M(4\rho) - \eta\omega(4\rho), 2\rho)|_{y^{1-2s}} \leq \Lambda |\mathscr{C}_{\Omega(z,R)}(Z, 2\rho)|_{y^{1-2s}}$,

where Λ *is determined by* (1.3.14) *with* $l = \frac{1}{2}$. *Then, there exists* $0 < \overline{\eta} < 1$ *independent of* Z *and* ρ *such that,*

$$\omega(\rho) \leq \overline{\eta}\omega(4\rho). \tag{1.3.22}$$

PROOF. Given $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ and $0 < 4\rho < \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$, let us define the sequence

$$k_j = M(4\rho) - \eta_j \omega(4\rho), \quad \text{with } \eta_j = \frac{1}{2^{j+1}}, \ j = 0, 1, \ldots$$

Assume first that $Z \in \mathscr{C}^\circ_{\Omega(z,R)} \backslash \Sigma^*_{\mathcal{D},R}$ so that $\mathcal{K}^+(Z) = (-\infty, \infty)$. Then, as $\lambda \leq 1/2$, one of the following conditions is satisfied,

$$\begin{gathered} |A^*_+(k_0, 2\rho)|_{y^{1-2s}} \leq (1-\lambda) |\mathscr{C}_{\Omega(z,R)}(Z, 2\rho)|_{y^{1-2s}} \\ \text{or} \\ |A^*_-(k_0, 2\rho)|_{y^{1-2s}} \leq (1-\lambda) |\mathscr{C}_{\Omega(z,R)}(Z, 2\rho)|_{y^{1-2s}}. \end{gathered} \tag{1.3.23}$$

Assume without loss of generality that $|A^*_+(k_0, 2\rho)| \leq (1-\lambda)|\mathscr{C}_{\Omega(z,R)}(Z, 2\rho)|$. As a consequence, $|A^*_+(k_j, 2\rho)| \leq (1-\lambda)|\mathscr{C}_{\Omega(z,R)}(Z, 2\rho)|$ for $j \geq 1$.
On the other hand, if $Z \in \Sigma^*_{\mathcal{D},R}$, we can assume that at least one of the numbers $M(4\rho)$ or $-m(4\rho)$ is greater than $\frac{1}{2}\omega(4\rho)$, suppose that $M(4\rho) > \frac{1}{2}\omega(4\rho)$. Therefore, $k_j > 0$ for $j \geq 0$. Then, using Lemma 1.3.6 with $h = k_{j+1}$ and $k = k_j$, we obtain

$$(k_{j+1} - k_j)^2 |A^*_+(k_{j+1}, 2\rho)|^{\frac{2}{q}}_{y^{1-2s}} \leq \beta_s^2 (2\rho)^2 |B_{2\rho}(Z)|^{2\left(\frac{1}{q} - \frac{1}{p}\right)}_{y^{1-2s}} \int_{A^*_+(k_j, 2\rho)} y^{1-2s} |\nabla W|^2 dxdy.$$

Moreover, applying Lemma 1.3.3 to the function $t^+_{k_j}(W) \in \mathcal{X}^s_{\Sigma_{\mathcal{D}},R}(\mathscr{C}_{\Omega(z,R)})$, $j \geq 0$, we find

$$\begin{aligned}\int_{A^*_+(k_j,2\rho)} y^{1-2s}|\nabla W|^2 dxdy \leq& \frac{C}{4\rho^2}\int_{A^*_+(k_j,4\rho)} y^{1-2s}|W-k_j|^2 dxdy\\ \leq& \frac{C}{4\rho^2}[M(4\rho)-k_j]^2|B_{4\rho}(Z)|_{y^{1-2s}}.\end{aligned}$$

As a consequence,

$$(1.3.24)\qquad \begin{aligned}(k_{j+1}-k_j)^2|A^*_+(k_{j+1},2\rho)|^{\frac{2}{q}}_{y^{1-2s}} \leq& C\beta_s|B_{2\rho}(Z)|^{2\left(\frac{1}{q}-\frac{1}{p}\right)+1}_{y^{1-2s}}[M(4\rho)-k_j]^2\\ &\times|A^*_+(k_j,2\rho)-A^*_+(k_{j+1},2\rho)|^{\frac{2}{p}-1}_{y^{1-2s}},\end{aligned}$$

with $C>0$ the constant appearing in the Caccioppoli inequality. Let us define

$$\varphi(k) = \frac{|A^*_+(k,2\rho)|_{y^{1-2s}}}{|\mathscr{C}_{\Omega(z,R)}(Z,2\rho)|_{y^{1-2s}}},$$

and note that, because of (1.3.10) and (1.3.11), we have $|B_{2\rho}(Z)|_{y^{1-2s}} \leq \frac{1}{\zeta_s}|\mathscr{C}_{\Omega(z,R)}(Z,2\rho)|_{y^{1-2s}}$. Then, as $2\left(\frac{1}{q}-\frac{1}{p}\right)+1>0$, taking into account that $k_{j+1}-k_j = \eta_{j+1}\omega(4\rho)$ and $M(4\rho)-k_j = \eta_j\omega(4\rho)$, from (1.3.24) we find

$$|\varphi(k_{j+1})|^{\frac{2}{q}} \leq \vartheta\left[\varphi(k_j)-\varphi(k_{j+1})\right]^{\frac{2}{p}-1},$$

with $\vartheta = \frac{4C\beta_s}{\zeta_s^{2\left(\frac{1}{q}-\frac{1}{p}\right)+1}}$. Let us set $\mu = \dfrac{2}{q}\dfrac{1}{\frac{2}{p}-1} > 0$ and $a = \frac{p}{2-p}$. Therefore,

$$|\varphi(k_n)|^{\mu} \leq \vartheta^a\left[\varphi(k_j)-\varphi(k_{j+1})\right],$$

for all $j \leq n$. Adding for $j = 0,1,\ldots,n$ and noticing that $\varphi(k_j) \geq \varphi(k_n)$ we get

$$n|\varphi(k_n)|^{\mu} \leq \vartheta^a\left[\varphi(k_0)-\varphi(k_{n+1})\right].$$

Hence, because of (1.3.23), we conclude

$$(1.3.25)\qquad |\varphi(k_n)| \leq \left(\frac{\vartheta^a\varphi(k_0)}{n}\right)^{\frac{1}{\mu}} \leq \left(\frac{\vartheta^a(1-\lambda)}{n}\right)^{\frac{1}{\mu}}.$$

Let us set $\overline{n}>0$ such that

$$(1.3.26)\qquad \overline{n} \geq \left\lceil\frac{\vartheta^a(1-\lambda)}{\Lambda^{\mu}}\right\rceil = \left\lceil\frac{(4C\beta_s)^a(1-\lambda)}{\zeta_s^{\mu-1}\Lambda^{\mu}}\right\rceil,$$

where Λ is determined by (1.3.14) with $l=\frac{1}{2}$, ζ_s depends on ζ in (1.3.10) and the A_2-*constant*(see (1.3.11)), the constant β_s depends on N and the weight y^{1-2s} and $C>0$ is an universal constant coming from the Caccioppoli inequality.
Consequently, $\overline{n}$ is independent of Z and ρ. Then, because of inequality (1.3.25), we find

$$\frac{|A^*_+(k_n,2\rho)|_{y^{1-2s}}}{|\mathscr{C}_{\Omega(z,R)}(Z,2\rho)|_{y^{1-2s}}} \leq \Lambda, \qquad \forall n \geq \overline{n}.$$

Applying Theorem 1.3.5 with $k_{\overline{n}} = M(4\rho) - \eta_{\overline{n}}\omega(4\rho)$, $r = 2\rho$ and $l = \frac{1}{2}$, so that

$$\frac{1}{\Lambda|B_{2\rho}(Z)|_{y^{1-2s}}}\int_{A^*_+(M(4\rho)-\eta_{\overline{n}}\omega(4\rho),2\rho)} y^{1-2s}|W-(M(4\rho)-\eta_{\overline{n}}\omega(4\rho))|^2dxdy \le (\eta_{\overline{n}}\omega(4\rho))^2 = d^2,$$

we obtain,

$$W(X) \le k + ld \le [M(4\rho)-\eta_{\overline{n}}\omega(4\rho)] + \frac{1}{2}\eta_{\overline{n}}\omega(4\rho) \le M(4\rho) - \frac{1}{2}\eta_{\overline{n}}\omega(4\rho),$$

a.e. in $\mathscr{C}_{\Omega(z,R)(Z,\rho)}$. As a consequence,

$$\begin{aligned}\omega(\rho) &= M(\rho) - m(\rho) \le M(\rho) - m(4\rho) \le [M(4\rho) - \frac{1}{2}\eta_{\overline{n}}\omega(4\rho)] - m(4\rho)\\ &\le (1-\frac{1}{2}\eta_{\overline{n}})\omega(4\rho).\end{aligned}$$

We conclude (1.3.22) with $\overline{\eta} = (1-\eta_{\overline{n}+1})$. □

THEOREM 1.3.8. *Given $z \in \overline{\Omega}$ and $R > 0$ assume that $\mathscr{C}_{\Omega(z,R)}$ is a λ-admissible set with $\lambda \le 1/2$ and $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ is a solution of the homogeneous problem* (1.3.4)*. Then, there exists $0 < \mathcal{H} < 1$, $0 < \tau < \frac{1}{2}$ and $\delta(Z) > 0$ such that for all $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$ and $0 < \rho < \delta(Z)$, we have*

$$\omega(\rho) = \sup_{X\in\overline{\mathscr{C}}_{\Omega(z,R)}(Z,\rho)} W(X) - \inf_{X\in\overline{\mathscr{C}}_{\Omega(z,R)}(Z,\rho)} W(X) \le \mathcal{H}\rho^\tau.$$

PROOF. Let $r(Z) = \min\{\widetilde{\rho}(Z), \overline{\rho}(Z)\}$. Because of Theorem 1.3.7, inequality (1.3.22) holds true for all $\rho < r(Z)/4$. Take τ, M positive such that $4^\tau\overline{\eta} = a < 1$ and $\omega(\rho) \le M\rho^\tau$ for $\frac{r(Z)}{4} \le \rho < r(Z)$. Then, because of (1.3.22), for $\frac{r(Z)}{4^2} \le \rho < \frac{r(Z)}{4}$ it holds

$$\omega(\rho) \le \overline{\eta}4^\tau M\rho^\tau.$$

In general, if $\frac{r(Z)}{4^{i+1}} \le \rho < \frac{r(Z)}{4^i}$, we conclude $\omega(\rho) \le (\overline{\eta}4^\tau)^i M\rho^\tau$. Letting $\overline{i}$ large enough such that $\mathcal{H} = Ma^{\overline{i}} < 1$, we obtain $\omega(\rho) \le \mathcal{H}\rho^\tau$ for all $\rho < \delta(Z) = \frac{r(Z)}{4^{\overline{i}}}$. On the other hand, since we have chosen $\tau > 0$ such that $4^\tau\overline{\eta} < 1$ and, because of Theorem 1.3.7, $\overline{\eta} = 1 - \eta_{\overline{n}+1}$ for some $\overline{n} \ge 0$ independent of Z and ρ, it follows that

$$\tau < \frac{1}{2}\log_2\left(\frac{2^{\overline{n}+2}}{2^{\overline{n}+2}-1}\right) < \frac{1}{2}. \tag{1.3.27}$$

□

To continue, let us observe the following:

(i) if $z \in \Omega$, then there exist $R > 0$ such that $\Sigma_{\mathcal{D},R} = \Sigma_{\mathcal{N},R} = \emptyset$. Hence, there is no Dirichlet nor Neumann part and $\widetilde{\rho}(Z) = dist(Z, \partial_L\mathscr{C}_{\Omega(z,R)})$ for all $Z \in \mathscr{C}_{\Omega(z,R)}$.

(ii) if $z \in \Sigma_{\mathcal{D}}\backslash\Gamma$, then there exist $R > 0$ such that $\Sigma_{\mathcal{N},R} = \emptyset$. Hence $\widetilde{\rho}(Z) = dist(Z, \partial_B\mathscr{C}_{\Omega(z,R)})$ for all $Z \in \Sigma^*_{\mathcal{D},R}$ and $\widetilde{\rho}(Z) = dist(Z, \partial_0\mathscr{C}_{\Omega(z,R)})$ for all $Z \in \mathscr{C}^\circ_{\Omega(z,R)}\backslash\Sigma^*_{\mathcal{D},R}$.

(iii) if $z \in \Sigma_{\mathcal{N}}$, then there exist $R > 0$ such that $\Sigma_{\mathcal{D},R} = \emptyset$. Hence we have $\widetilde{\rho}(Z) = dist(Z, \partial_B\mathscr{C}_{\Omega(z,R)})$ for all $Z \in \mathscr{C}^\circ_{\Omega(z,R)}$.

(iv) if $z \in \Gamma$ there is always a Dirichlet part and a Neumann part and hence $\widetilde{\rho}(Z) = dist(Z, \partial_B\mathscr{C}_{\Omega(z,R)})$ for all $Z \in \Sigma^*_{\mathcal{D},R}$ and $\widetilde{\rho}(Z) = dist(Z, \partial_0\mathscr{C}_{\Omega(z,R)})$ for all $Z \in \mathscr{C}^\circ_{\Omega(z,R)}\backslash\Sigma^*_{\mathcal{D},R}$.

Now, consider $\overline{\mathscr{C}}_{\Omega(z,R/2)} \subset \mathscr{C}_{\Omega(z,R)}$ if $z \in \Omega$ and $\overline{\mathscr{C}}_{\Omega(z,R/2)} \subset \mathscr{C}^{\circ}_{\Omega(z,R)}$ if $z \in \partial\Omega$. Hence it is clear that

(i) if $z \in \Omega$, then $\widetilde{\rho}(Z) = dist(Z, \partial_L \mathscr{C}_{\Omega(z,R)}) \geq \widetilde{\rho} > 0$ for all $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$ and some positive $\widetilde{\rho}$.

(ii) if $z \in \Sigma_{\mathcal{D}} \backslash \Gamma$, then $\widetilde{\rho}(Z) = \widetilde{\rho} > 0$ for some positive $\widetilde{\rho}$ for all $Z \in \Sigma^*_{\mathcal{D},R/2}$ and $\widetilde{\rho}(Z) = dist(Z, \Sigma^*_{\mathcal{D},R/2})$ for all $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)} \backslash \Sigma^*_{\mathcal{D},R/2}$.

(iii) if $z \in \Sigma_{\mathcal{N}}$, then $\widetilde{\rho}(Z) = dist(Z, \partial_B \mathscr{C}_{\Omega(z,R)}) \geq \widetilde{\rho} > 0$ for all $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$ and some positive $\widetilde{\rho}$.

(iv) if $z \in \Gamma$ then $\widetilde{\rho}(Z) = \widetilde{\rho} > 0$ for some positive $\widetilde{\rho}$ for all $Z \in \Sigma^*_{\mathcal{D},R/2}$ and $\widetilde{\rho}(Z) = dist(Z, \Sigma_{\mathcal{D},R/2})$ for all $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)} \backslash \Sigma^*_{\mathcal{D},R/2}$.

In the situation of items (i) and (iii), if we assume that $\mathscr{C}_\Omega$ is an uniform λ-admissible set, then the number $0 < \delta(Z)$ in Theorem 1.3.8 has an infimum value, namely $0 < \delta < \delta(Z)$ for all $Z \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$ and we deduce that solutions W to problem (1.3.4) are Hölder continuous up to the boundary of $\mathscr{C}_{\Omega(z,R/2)}$. In fact, let us consider two points Z_1 and Z_2 in $\overline{\mathscr{C}^m_{\Omega(z,R)}}$ with $m > 0$. Then, because of Corollary 1.3.1 and Theorem 1.3.8 we find

- If $|Z_1 - Z_2| \geq \delta$, we have
$$\frac{|W(Z_1) - W(Z_2)|}{|Z_1 - Z_2|^\tau} \leq \frac{2}{\delta^\tau} \max_{\mathscr{C}^m_{\Omega(z,R/2)}} W = \frac{2}{\delta^\tau} \|W\|_{L^\infty(\mathscr{C}^m_{\Omega(z,R/2)})}.$$
- If $|Z_1 - Z_2| < \delta$, by Theorem 1.3.8, $\frac{|W(Z_1)-W(Z_2)|}{|Z_1-Z_2|^\tau} \leq \mathcal{H}$, $0 < \mathcal{H} < 1$.

We conclude the Hölder regularity with a constant

$$\mathcal{T} = \max\{\mathcal{H}, \frac{2}{\delta^\tau} \|W\|_{L^\infty(\mathscr{C}^m_{\Omega(z,R/2)})}\}. \tag{1.3.28}$$

Now we deal with the situation described in items (ii) and (iv).

THEOREM 1.3.9. *Given $z \in \Sigma_{\mathcal{D}}$ and $R > 0$, assume that $\mathscr{C}_{\Omega(z,R)}$ is an uniform λ-admissible set with $\lambda \leq 1/2$ and $W \in \mathcal{X}^s_{\Sigma_{\mathcal{D},R}}(\mathscr{C}_{\Omega(z,R)})$ is a solution of the homogeneous problem* (1.3.4)*. Then, the function $W \in \mathcal{C}^\tau_{loc}(\overline{\mathscr{C}}_{\Omega(z,R/2)})$ for some $0 < \tau < \frac{1}{2}$.*

PROOF. Since $\mathscr{C}_{\Omega(z,R)}$ is an uniform λ-admissible set, the number $0 < \delta(Z)$ in Theorem 1.3.8 is bounded from below by some $0 < \delta_{\mathtt{H}}$ for $Z \in \Sigma^*_{\mathcal{D},R/2}$ and we can assume that $\delta(Z) \geq \min\left\{\delta_{\mathtt{H}}, dist(Z, \Sigma^*_{\mathcal{D},R/2})\right\}$ for $Z \in \Sigma^*_{\mathcal{N},R/2}$. Moreover, because of the construction of the lateral boundary of the extension cylinder, the numbers $\delta(Z)$ do not depend on the y variable. Hence this infimum value $\delta_{\mathtt{H}} > 0$ is determined by the points $Z = (z, 0)$ in $\partial\Omega \times \{0\}$. Consider the set

$$\mathscr{C}^\delta_{\Omega(z,R/2)} = \{Z \in \overline{\mathscr{C}^m_{\Omega(z,R/2)}} : \ dist(Z, \Sigma^*_{\mathcal{D},R/2}) \geq \delta_{\mathtt{H}}\}.$$

As above, we only need to study the case $|Z_1 - Z_2| < \delta_{\mathtt{H}}$. Suppose that $Z_1 \in \mathscr{C}^\delta_{\Omega(z,R/2)}$, then $|Z_1 - Z_2| \leq \delta_{\mathtt{H}} < dist(Z_1, \Sigma^*_{\mathcal{D},R/2}) = \delta(Z_1)$, and thus, because of Theorem 1.3.8, we have

$$\frac{|W(Z_1) - W(Z_2)|}{|Z_1 - Z_2|^\tau} \leq \mathcal{H}.$$

If neither Z_1 nor Z_2 belongs to $\mathscr{C}^{\delta}_{\Omega(z,R/2)}$ but one of them, for example $Z_1 \in \Sigma^*_{\mathcal{D},R/2}$, we have $|Z_1 - Z_2| \leq \delta_{\mathtt{H}} = \delta(Z_1)$, and the results follows as before. If, instead, none of them belongs neither to $\mathscr{C}^{\delta}_{\Omega(z,R/2)}$ nor to $\Sigma^*_{\mathcal{D},R/2}$, we have two cases:

- $|Z_1 - Z_2| \leq \max\Big\{dist(Z_1, \Sigma^*_{\mathcal{D},R/2}), dist(Z_2, \Sigma^*_{\mathcal{D},R/2})\Big\}$.
- $|Z_1 - Z_2| > \max\Big\{dist(Z_1, \Sigma^*_{\mathcal{D},R/2}), dist(Z_2, \Sigma^*_{\mathcal{D},R/2})\Big\}$.

In the first case at least one of the two points, say Z_1, satisfies the inequality $|Z_1 - Z_2| \leq \delta_{\mathtt{H}} < dist(Z_1, \Sigma^*_{\mathcal{D},R/2}) = \delta(Z_1)$ and we have the result as before. In the second case, there exists at least one $\overline{Z} \in \Sigma^*_{\mathcal{D},R/2}$ such that $|\overline{Z} - Z_1| \leq |Z_1 - Z_2|$, and using the triangle inequality it follows that $|\overline{Z} - Z_2| \leq 2|Z_1 - Z_2|$. Since the result has been proved for the case when at least one point belongs to $\Sigma^*_{\mathcal{D},R/2}$, we find

$$|W(Z_1) - W(Z_2)| \leq |W(Z_1) - W(\overline{Z})| + |W(\overline{Z}) - W(Z_2)| \leq 3\mathcal{H}|Z_1 - Z_2|^{\tau}, \tag{1.3.29}$$

and we conclude the Hölder regularity with constant $\mathcal{T} = \max\{3\mathcal{H}, 2\delta_{\mathtt{H}}^{-\tau}\|W\|_{L^\infty(\mathscr{C}_{\Omega(z,R/2)})}\}$, with $0 < \mathcal{H} < 1$ given by Theorem 1.3.8, see (1.3.28). $\square$

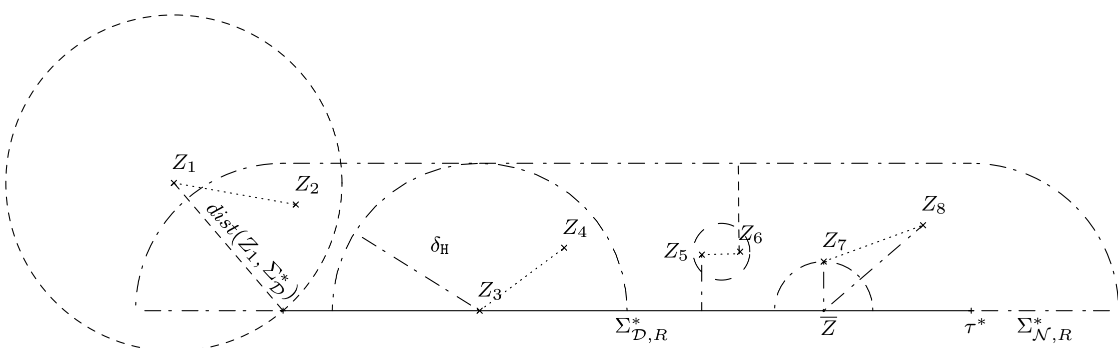

FIGURE 1. The different cases presented in the proof of Theorem 1.3.9.

COROLLARY 1.3.2. *Let Ω be a smooth domain such that $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ satisfy hypotheses* ($\mathfrak{B}$) *and let w be the solution of problem* (1.3.2) *with $z \in \overline{\Omega}$ and $R > 0$. Then, the function $w \in \mathcal{C}^{\tau}(\overline{\Omega(z, R/2)})$ for some $0 < \tau < \frac{1}{2}$.*

PROOF. Since Ω is a smooth bounded domain and the boundary manifolds satisfy hypotheses ($\mathfrak{B}$), then, for any $z \in \overline{\Omega}$, the extension cylinder $\mathscr{C}_{\Omega(z,R)}$ is an uniform λ-admissible set for any $0 < \lambda < 1$. Hence, there exists $0 < \delta_{\mathtt{H}} < \delta(Z)$ for $Z \in \Sigma^*_{\mathcal{D},R/2}$ and we can assume that $\delta(Z) \geq \min\Big\{\delta_{\mathtt{H}}, dist(Z, \Sigma^*_{\mathcal{D},R/2})\Big\}$ for $Z \in \Sigma^*_{\mathcal{N},R/2}$, with $\delta(Z)$ given in Theorem 1.3.8. Suppose that $z_1, z_2 \in \overline{\Omega}$:

- If $|z_1 - z_2| \geq \delta_{\mathtt{H}}$. Then, due to Corollary 1.3.1 we have $\|w\|_{L^\infty(\Omega(z,R/2))} < \infty$ and, therefore,
$$\frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^{\tau}} \leq \frac{2}{\delta_{\mathtt{H}}^{\tau}} \max_{\Omega(z,R/2)} w.$$

- While for $|z_1 - z_2| < \delta_{\mathtt{H}}$, let us set $Z_1 = (z_1, 0)$ and $Z_2 = (z_2, 0)$, $Z_1, Z_2 \in \overline{\mathscr{C}}_{\Omega(z,R/2)}$, such that $|Z_1 - Z_2| < \delta_{\mathtt{H}}$. Then, as in (1.3.29) in Theorem 1.3.9,

$$\frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^\tau} = \frac{|W(Z_1) - W(Z_2)|}{|Z_1 - Z_2|^\tau} \leq 3\mathcal{H}, \quad 0 < \mathcal{H} < 1.$$

Hence, we conclude

$$|w(z_1) - w(z_2)| \leq \mathcal{T}|z_1 - z_2|^\tau, \quad \forall z_1, z_2 \in \overline{\Omega(z, R/2)},$$

with $\mathcal{T} = \max\{3\mathcal{H}, 2\delta_{\mathtt{H}}^{-\tau}\|w\|_{L^\infty(\Omega(z,R/2))}\}$, and $\delta_{\mathtt{H}} > 0$ given as in Theorem 1.3.9. □

We prove now the main result of this work.

Proof of Theorem 1.1.1. Let u be the solution of problem (P^s), Ω a smooth bounded domain such that $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ satisfy hypotheses $(\mathfrak{B})$ and $f \in L^p(\Omega)$ for $p > \frac{N}{2s}$. Given $z \in \overline{\Omega}$ and $0 < R < 1$, let v be the solution to (1.3.1) and $w = u - v$ a function satisfying (1.3.2). Thus, using (1.3.13) and Corollary 1.3.2, we conclude that, for any $x, y \in \overline{\Omega(z, R/2)}$,

$$\omega(u, R/2) \leq \omega(w, R/2) + 2 \max_{x \in \Omega(z,R/2)} v(x) \leq \mathcal{T}R^\tau + C(N, s, |\Sigma_{\mathcal{D}}|)||f||_p R^{2s - \frac{N}{p}} \leq \mathcal{C}R^\gamma,$$

where $\gamma = \min\{\tau, 2s - \frac{N}{p}\} < \frac{1}{2}$ and $\mathcal{C} = \max\{\mathcal{T}, 2C(N, s, |\Sigma_{\mathcal{D}}|)||f||_p\}$, with

$$\mathcal{T} = \max\{3\mathcal{H}, 2\delta_{\mathtt{H}}^{-\tau}\|w\|_{L^\infty(\Omega(z,R/2))}\} = \max\{3\mathcal{H}, 2\delta_{\mathtt{H}}^{-\tau}\|u - v\|_{L^\infty(\Omega(z,R/2))}\}.$$

Moreover, because of Theorem 1.3.4, $\|u - v\|_{L^\infty(\Omega(z,R/2))} \leq \|u\|_{L^\infty(\Omega(z,R))} + \|v\|_{L^\infty(\Omega(z,R))} \leq C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_p$ hence we obtain

$$\mathcal{T} \leq \max\{3\mathcal{H}, 4\delta_{\mathtt{H}}^{-\tau}C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_p\}.$$

Therefore, $\mathcal{C} = \max\{3\mathcal{H}, 4\delta_{\mathtt{H}}^{-\tau}C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_p\}$. Repeating the steps above in Theorem 1.3.9, we conclude

$$|u(x) - u(y)| \leq \mathscr{H}|x - y|^\gamma, \text{ for all } x, y \in \overline{\Omega(z, R/2)}, \tag{1.3.30}$$

where

$$\mathscr{H} = \max\{9\mathcal{H}, \frac{C(N, s, |\Sigma_{\mathcal{D}}|)\|f\|_p}{\delta_{\mathtt{H}}^\gamma}\},$$

and $\gamma = \min\{\tau, 2s - \frac{N}{p}\} < \frac{1}{2}$. Since the constants $\mathscr{H}$ and γ do not depend on z nor R, to complete the proof, set $z_i \in \overline{\Omega}$, $i = 1, 2, \ldots, m$ and $R_i > 0$, small enough such that

$$\overline{\Omega} = \bigcup_{i=1}^m \Omega(z_i, R_i/4).$$

Then, given $x, y \in \overline{\Omega}$, we can assume that $x, y \in \Omega(z_i, R_i/2)$ for some $i \geq 1$ and, hence, we conclude (1.3.30) in $\overline{\Omega}$. □

1.4. Moving the boundary conditions

In this last part, we study the behavior of the solutions to problem (P^s) when we move the boundary conditions. First, let us describe this mixed moving boundary data framework. As introduced above, given $I_\varepsilon = [\varepsilon, |\partial\Omega|]$, let us consider the family of closed sets $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$, satisfying

(B_1) $\Sigma_{\mathcal{D}}(\alpha)$ has a finite number of connected components.
(B_2) $\Sigma_{\mathcal{D}}(\alpha_1) \subset \Sigma_{\mathcal{D}}(\alpha_2)$ if $\alpha_1 < \alpha_2$.
(B_3) $|\Sigma_{\mathcal{D}}(\alpha_1)| = \alpha_1 \in I_\varepsilon$.

We call $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega\backslash\Sigma_{\mathcal{D}}(\alpha)$ and $\Gamma(\alpha) = \Sigma_{\mathcal{D}}(\alpha)\cap\overline{\Sigma}_{\mathcal{N}}(\alpha)$. Observe that, under the hypotheses (B_1)–(B_3), the limit sets $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$ as $\alpha \to \varepsilon^+$ are not degenerated sets (for instance a Cantor-like set).
For a family of this type we consider the corresponding family of mixed boundary value problems

$$(P^s_\alpha) \qquad \begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $B_\alpha(u)$ means $B(u)$ with $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$, and Γ are replaced by $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$, and $\Gamma(\alpha)$ respectively. Similarly, $(\mathfrak{B}_\alpha)$ means $(\mathfrak{B})$ with the natural changes as above.

In this scenario we prove the following result.

Corollary 1.4.1. *Suppose that $0 < \varepsilon < |\partial\Omega|$ and consider the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$ satisfies the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)–(B_3) and let u_α be a solution of (P^s_α). Then, there exists two constants $0 < \gamma < \frac{1}{2}$ and $\mathscr{H}_\varepsilon > 0$ independent of $\alpha \in [\varepsilon, |\partial\Omega|]$ such that*

$$\|u\|_{\mathcal{C}^\gamma(\Omega)} \le \mathscr{H}_\varepsilon.$$

The key point to obtain Corollary 1.4.1 is to prove that we can choose $\beta_s > 0$ in (1.3.9) independent of the measure of the Dirichlet part. Nevertheless, as we will see below, when one takes $\alpha \to 0^+$ the control of the Hölder norm of such a family is lost. Hence, it is necessary to fix a positive minimum $\varepsilon > 0$ on the measure of the family $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in I_\varepsilon}$, in order to guarantee the control on the Hölder norm for the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$.

Proof of Corollary 1.4.1. Assume that $\partial\Omega$ is a smooth manifold and $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$ satisfy hypotheses $(\mathfrak{B})$, i.e. $\mathscr{C}_\Omega$ is an uniform λ-admissible set for any $0 < \lambda < 1$. Thus, there exists $\delta > 0$ such that $\overline{\rho}(Z) \ge \delta$ for all $Z \in \partial_L\mathscr{C}_\Omega$. Then, taking in mind (i^*)-(ii^*) in Remark 1.3.1, we have the following.

(1) If $Z \in \overline{\mathscr{C}}_\Omega\backslash\Sigma^*_{\mathcal{D}}(\alpha)$, inequality (1.3.9) holds true with $\beta_s = \frac{c_s}{\zeta\lambda}$ independent of α, for all $0 < \rho < \delta$.
(2) If $Z \in \Sigma^*_{\mathcal{D}}(\alpha) \setminus \Gamma^*(\alpha)$, we can set $0 < \rho < \min\{\delta, dist(Z, \Gamma^*(\alpha))\}$, such that for all $X \in \mathscr{C}_\Omega(Z, \rho)$,
$$\Pi(X, \Sigma^*_{\mathcal{D}} \cap B_\rho(Z), \mathscr{C}_\Omega(Z, \rho)) \ge \varphi > 0,$$
with φ independent of α. Hence, inequality (1.3.9) holds true with $\beta_s \le \frac{c_s}{\varphi}$ also independent of α.
(3) If $Z \in \Gamma^*(\alpha)$, we can assume without loss of generality that, for some neighborhood of radius $0 < \rho < \min\{\delta, \delta_\Gamma\}$ of the point $Z = (Z_1, \dots, Z_{N+1})$, $\partial_L\mathscr{C}_\Omega$ coincides with the hyperplane $\mathbb{R}^{N+1} \cap \{x_N = 0\}$ and $\Gamma^*(\alpha) \subset \mathbb{R}^{N+1}_+ \cap \{x_N = 0, x_{N-1} = 0\}$, in such

a way that in $\Sigma^*_{\mathcal{D}}(\alpha)$ we have $x_{N-1} \geq 0$ and, in $\Sigma^*_{\mathcal{N}}(\alpha)$ we have $x_{N-1} < 0$. Now, $\mathscr{C}_\Omega(Z,\rho)$ is transformed by the bi-Lipschitz transform[6],

$$x_i = \xi_i, \quad , i = 1, 2, \dots, N-1,$$

$$x_N = \begin{cases} \xi_N & \text{if } \xi_{N-1} < 0, \\ \xi_N - \xi_{N-1} & \text{if } \xi_{N-1} \geq 0, \end{cases}$$

into a set $\mathcal{O}_\rho(Z) = \mathcal{O}^1_\rho(Z) \cup \mathcal{O}^2_\rho(Z)$ with

$$\mathcal{O}^1_\rho(Z) = \left\{ \xi_N \geq 0, \xi_{N-1} < 0, \sum_{i=1}^{N} (\xi_i - Z_i)^2 + (y - Z_{N+1})^2 \leq \rho^2 \right\},$$

$$\mathcal{O}^2_\rho(Z) = \left\{ \begin{array}{l} \xi_{N-1} \geq 0, \displaystyle\sum_{i=1}^{N-1} (\xi_i - Z_i)^2 + (y - Z_{N+1})^2 \leq \rho^2, \\ \xi_{N-1} \leq \xi_N \leq \xi_{N-1} + \left(\rho^2 - \displaystyle\sum_{i=1}^{N-1} (\xi_i - Z_i)^2 - (y - Z_{N+1})^2 \right)^{\frac{1}{2}} \end{array} \right\}.$$

While $\Sigma^*_{\mathcal{D}} \cap B_\rho(Z)$ is transformed into the set

$$\mathcal{D}_\rho(Z) = \left\{ \xi_N = \xi_{N-1}, \xi_{N-1} \geq 0, \sum_{i=1}^{N-1} (\xi_i - Z_i)^2 - (y - Z_{N+1})^2 \leq \rho^2 \right\}.$$

Given $X_0 \in \mathcal{O}_\rho(Z)$, we continue using the representation, see [**80**, cfr. 13.1],

$$\Pi(X_0, \mathcal{D}_\rho(Z), \mathcal{O}_\rho(Z)) = \frac{1}{|\mathbb{S}_N(X_0)|} \int_{\mathcal{D}_\rho(Z)} \frac{1}{|X_0 - Y|^N} \cos(\psi) d\sigma,$$

where $\cos(\psi) = \langle \frac{X_0 - Y}{|X_0 - Y|}, \vec{v} \rangle$, with $\vec{v}$ the normal vector to $\{\xi_N = \xi_{N-1}\} \cap \mathbb{R}^{N+1}_+$. Since $\cos(\psi)$ vanish only when $X_0 \in \mathcal{D}_\rho(Z)$ we conclude that $\Pi(X_0, \mathcal{D}_\rho(Z), \mathbb{R}^{N+1}_+) \geq \varphi > 0$ for all $X_0 \in \mathcal{O}_\rho(Z)$ and some $\varphi > 0$ independent of α. On the other hand, it is immediate that φ is independent of ρ. Hence, inequality (1.3.9) holds true with $\beta_s \leq \frac{c_s}{\varphi}$ also independent of α.

Let us define

$$\overline{\rho}_\alpha(Z) := \begin{cases} \min\{\delta, dist(Z, \Sigma^*_{\mathcal{D}})\}, & \text{if } Z \in \overline{\mathscr{C}}_\Omega \backslash \Sigma^*_{\mathcal{D}}(\alpha), \\ \min\{\delta, dist(Z, \Gamma^*)\}, & \text{if } Z \in \Sigma^*_{\mathcal{D}}(\alpha) \setminus \Gamma^*(\alpha), \\ \min\{\delta, \delta_\Gamma\}, & \text{if } Z \in \Gamma^*(\alpha). \end{cases} \tag{1.4.1}$$

As a consequence of (1)–(3) above, we conclude

(i) Because of (1.3.19), the constant Λ appearing in Theorem 1.3.5, Theorem 1.3.6 and Theorem 1.3.7, is independent of α. Hence, inequality (1.3.21) does not depends on α. Thus, the number $0 < \mathcal{H} < 1$ in Theorem 1.3.8 is independent of α.
(ii) Because of (1.3.26), the constant $\overline{\eta}$ in Theorem 1.3.7 is independent of α. Moreover, because of (1.3.27), we conclude that $0 < \gamma < \frac{1}{2}$ is independent of α.

[6]We only perform a transformation in the $x_1, \dots, x_N$ variables, without change the extension variable y.

Then, given u_α a solution of problem (P^s_α) with $\alpha \in I_\varepsilon$, because of Theorem 1.1.1, we conclude

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\Omega)} \leq \mathscr{H}_\alpha,$$

with $\gamma = \min\{\tau, 2s - \frac{N}{p}\} < \frac{1}{2}$ independent of α and $\mathscr{H}_\alpha = \max\{9\mathcal{H}, \frac{C(N,s,\alpha)\|f\|_p}{\delta^\tau_{\mathtt{H},\alpha}}\}$ with the constants $0 < \tau < \frac{1}{2}$ and $\delta_{\mathtt{H},\alpha}$ given as in Corollary 1.3.2. Now, if we consider the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$, since $\overline{\rho}_{\alpha_1}(Z) \leq \overline{\rho}_{\alpha_2}(Z)$ it is clear that $\delta_{\mathtt{H},\alpha_1} \leq \delta_{\mathtt{H},\alpha_2}$ and, therefore, $\mathscr{H}_{\alpha_1} \geq \mathscr{H}_{\alpha_2}$ for all $\alpha_1, \alpha_2 \in [\varepsilon, |\partial\Omega|]$, $\alpha_1 \leq \alpha_2$. Therefore, we can take $0 < \gamma < \frac{1}{2}$ and $\mathscr{H}_\varepsilon = \max\{9\mathcal{H}, \frac{C(N,s,\varepsilon)\|f\|_p}{\delta^\tau_{\mathtt{H},\varepsilon}}\}$ independent of α such that

$$\|u_\alpha\|_{\mathcal{C}^\gamma(\Omega)} \leq \mathscr{H}_\varepsilon,$$

To conclude, we observe that the condition $\alpha \in [\varepsilon, |\partial\Omega|]$ is necessary in order to control the Hölder norm of the family $\{u_\alpha\}_{\alpha\in I_\varepsilon}$. If we let $\alpha = |\Sigma_{\mathcal{D}}(\alpha)| \to 0^+$, then it is clear that $|\Sigma^*_{\mathcal{D}}(\alpha) \cap \overline{\mathscr{C}}_\Omega(Z,\rho)| \to 0$ for any $Z \in \overline{\mathscr{C}}_\Omega$ and $\rho > 0$. Thus, if $\alpha \to 0^+$, we conclude from (1.4.1) that $\overline{\rho}_\alpha(Z) \to 0$ for any $Z \in \Sigma^*_{\mathcal{D}}$ and, hence, $\delta_{\mathtt{H},\alpha} \to 0$ as $\alpha \to 0^+$. As a consequence, $\mathscr{H}_\alpha \to +\infty$ as $\alpha \to 0^+$ and the control on the Hölder norm is lost. □

REMARK 1.4.1. *Given an interphase point* $Z \in \Gamma^*$*, it is clear from* (1.4.1)*, that we can choose an uniform* $\rho_\varepsilon > 0$ *in the lines of* [**34**, Corollary 6.1]*. In fact, it is enough to choose* δ_Γ *in* (1.4.1) *in such a way that* $\Sigma^*_{\mathcal{D}}(\varepsilon) \cap \overline{\mathscr{C}}_\Omega(Z,\rho)$ *is contained in some hyperplane (see (3) in the proof of Corollary 1.4.1). Clearly, this Dirichlet boundary part, say* $\Big(\{x_N = 0, x_{N-1} \geq 0\} \cap \mathbb{R}^{N+1}_+\Big) \cap B_{\rho_\varepsilon}(Z)$*, converges to an empty set as* $\rho_\varepsilon \to 0$ *and the control on the Hölder norm is lost.*

CHAPTER 2

Semilinear fractional elliptic problems with mixed Dirichlet-Neumann boundary conditions

We conclude the main pourpose of **Part I** with the contents of Chapter 2, where we study a nonlinear elliptic problem defined on a bounded domain involving the fractional Laplace operator, a concave-convex term together with mixed boundary conditions.

2.1. Introduction

We study a nonlinear elliptic problem involving the fractional Laplace operator and a concave-convex power term together with mixed Dirichlet-Neumann boundary conditions. Namely,

$$(P_\lambda^s) \qquad \left\{\begin{aligned} (-\Delta)^s u &= \lambda u^q + u^r && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Sigma_{\mathcal{D}}, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Sigma_{\mathcal{N}}, \end{aligned}\right.$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with smooth boundary, $(-\Delta)^s$, with $\frac{1}{2} < s < 1$, denotes the spectral fractional Laplace operator, $\lambda > 0$ is a real parameter and $0 < q \leq 1 < r < \frac{N+2s}{N-2s}$. In order to simplify the notation we denote the mixed boundary conditions as

$$B(u) = \chi_{\Sigma_{\mathcal{D}}}(x) \cdot u + \chi_{\Sigma_{\mathcal{N}}}(x) \cdot \frac{\partial u}{\partial \nu}, \tag{2.1.1}$$

where χ_A stands for the characteristic function of a set A and we assume that the boundary manifolds $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are such that

$$(\mathfrak{B}) \quad \left\{\begin{array}{l} \Sigma_{\mathcal{D}} \text{ and } \Sigma_{\mathcal{N}} \text{ are smooth } (N-1)\text{-dimensional submanifolds of } \partial\Omega. \\ \Sigma_{\mathcal{D}} \text{ is a closed manifold of positive } (N-1)\text{-dimensional Lebesgue measure,} \\ \qquad\qquad |\Sigma_{\mathcal{D}}| = \alpha \in (0, |\partial\Omega|). \\ \Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset\,,\ \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega \text{ and } \Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma \text{ where } \Gamma \text{ is a smooth} \\ (N-2)\text{-dimensional submanifold of } \partial\Omega. \end{array}\right.$$

Problems like (P_λ^s) have been studied in the last decades: with the classical Laplace operator and Dirichlet boundary condition, c.f. [**65**] or [**10**] for a deep study; with the Laplace operator and mixed Dirichlet-Neumann boundary conditions, c.f. [**1, 2, 34**]; with the p-Laplace operator, c.f. [**23, 54, 55**]; with fully nonlinear operators, c.f. [**32**]; and more recently with the fractional Laplace operator and Dirichlet boundary conditions, c.f. [**18, 19, 24**]. Up to our knowledge, this is the first work where the concave-convex problem

is analyzed with the spectral fractional Laplace operator associated with mixed Dirichlet-Neumann boundary conditions.
The main result to be proven in this chapter is stated as follows.

THEOREM 2.1.1. *Assume that* $\frac{1}{2} < s < 1$, $N > 2s$ *and* $0 < q \leq 1 < r < \frac{N+2s}{N-2s}$. *Then*

(1) If $q = 1$ *there exists at least one solution of* (P_λ) *for every* $0 < \lambda < \lambda_{1,s}$, *where* $\lambda_{1,s}$ *denotes the first eigenvalue of the spectral fractional Laplacian with the boundary conditions* (2.1.1), *while there is no solution for* $\lambda \geq \lambda_{1,s}$. *Even more, there is a branch of solutions to* (P_λ) *bifurcating from* $(\lambda_{1,s}, 0)$, *which cuts the axis* $\{\lambda = 0\}$.

(2) If $0 < q < 1$ *there exists* $0 < \Lambda < \infty$ *such that:*
  - *(a) If* $0 < \lambda < \Lambda$ *there is a minimal solution of* (P_λ^s). *Moreover, the family of minimal solutions is increasing with respect to* λ.
  - *(b) If* $\lambda = \Lambda$ *there is at least one solution of* (P_λ^s).
  - *(c) If* $\lambda > \Lambda$ *there is no solution of* (P_λ^s).
  - *(d) Problem* (P_λ^s) *admits at least two solutions for every* $0 < \lambda < \Lambda$.

The next result deals with the sub-linear case $0 < q < 1$ and provides us with uniform $L^\infty(\Omega)$-bounds for all the solutions to problems (P_λ^s) for any $0 < \lambda \leq \Lambda$.

THEOREM 2.1.2. *There exists a positive constant* $C = C(N, s, \Omega, r, q)$ *such that any solution* u_λ *to problem* (P_λ^s) *with* $\frac{1}{2} < s < 1$, $N > 2s$, $0 < q < 1 < r < \frac{N+2s}{N-2s}$ *and* $\lambda \in (0, \Lambda]$ *satisfies*

$$\sup_{x \in \Omega} u_\lambda(x) \leq C\,.$$

We also obtain uniform L^∞-estimates, in the case in which we move the boundary conditions. To be precise we consider a family of sets $\{\Sigma_{\mathcal{D}}(\alpha)\}$, with $\alpha \in (0, |\partial\Omega|]$ and $|\cdot|$ denoting the Lebesgue measure in the appropriate dimension, such that:

(B_1) $\Sigma_{\mathcal{D}}(\alpha)$ is connected or has a finite number of connected components.
(B_2) $\Sigma_{\mathcal{D}}(\alpha_1) \subset \Sigma_{\mathcal{D}}(\alpha_2)$ if $\alpha_1 < \alpha_2$.
(B_3) $|\Sigma_{\mathcal{D}}(\alpha)| = \alpha$.

We call $\Sigma_{\mathcal{N}}(\alpha) = \partial\Omega \backslash \Sigma_{\mathcal{D}}(\alpha)$ and we assume that $\Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha) = \Gamma(\alpha)$ is a $(N-2)$-dimensional smooth submanifold. For a family of this type we consider the corresponding family of mixed boundary value problems,

$$(P_{\alpha,\lambda}^s) \qquad \left\{ \begin{array}{ll} (-\Delta)^s u = \lambda u^q + u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B_\alpha(u) = 0 & \text{on } \partial\Omega, \end{array} \right.$$

where $B_\alpha(u)$ is defined as $B(u)$ with $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ replaced by $\Sigma_{\mathcal{D}}(\alpha)$, $\Sigma_{\mathcal{N}}(\alpha)$ satisfying the corresponding hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3). In this scenario we prove the following result.

THEOREM 2.1.3. *Consider the family* $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha \in (0, |\partial\Omega|]}$ *satisfying the hypotheses* $(\mathfrak{B}_\alpha)$ *and* (B_1)-(B_3). *For every* $0 < \varepsilon < |\partial\Omega|$, *let us denote* $I_\varepsilon = [\varepsilon, |\partial\Omega|]$ *and let*

$$\mathcal{S}_\varepsilon = \{u : \Omega \to \mathbb{R} |\ \textit{such that } u \textit{ is solution of } (P_{\alpha,\lambda}^s),\ \textit{with } \alpha \in I_\varepsilon\}.$$

Then, there exists a constant $\mathcal{M}_\varepsilon > 0$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq \mathcal{M}_\varepsilon, \quad \forall u \in \mathcal{S}_\varepsilon.$$

In addition, we will also prove the following behavior for the minimal solutions as we move the boundary conditions:

THEOREM 2.1.4. *Consider the family* $\{\Sigma_{\mathcal{D}}(\alpha)\}_{\alpha\in(0,|\partial\Omega|]}$ *satisfying the hypotheses* $(\mathfrak{B}_\alpha)$ *and* (B_1)-(B_3). *Then*

(1) *the minimal solutions* $\{\underline{u}(\alpha)\}$ *are uniformly bounded for any* $\alpha\in[0,|\partial\Omega|]$. *Moreover,*

$$\|\underline{u}(\alpha)\|_{H^s(\Omega)},\ \|\underline{u}(\alpha)\|_{L^\infty(\Omega)}\to 0 \ \text{ as } \alpha\to 0;$$

(2) *the non minimal solutions (of mountain pass type) are bounded and they converge to zero in* $H^s(\Omega)$ *as* $\alpha\to 0$.

Chapter 2 is organized as follows: In Section 2.2 we introduce the appropriate functional framework and some definitions needed in the sequel. In Section 2.3 we study a half-space problem that will be useful in the proof of the main theorem. To this end we make use of the moving planes method and we extend some results of [**37**] to the fractional setting. Section 2.4 is devoted to the study of the concave-convex problem by means of certain limit problems. This section contains the proof of Theorem 2.1.2 and Theorem 2.1.3 which are based on the blow-up method of [**60**]. To accomplish this step we need some compactness properties that requires to know precise Hölder estimates for the solutions to mixed boundary problems. We use the results of Chapter 1 where the Hölder regularity of such solutions is proven. Section 2.5 is devoted to the proof of Theorem 2.1.1 and Theorem 2.1.4 about the behavior when we move the boundary conditions of some class of solutions.

2.2. Functional setting and preliminaries

Along this chapter we follow the notation and the functional framework of Chapter 1, so that we refer to Section 1.2 for the definition of the spectral fractional Laplacian as well as the useful properties about the extension technique exposed there.
Since the definition of the spectral fractional Laplacian given in Section 1.2 allows us to integrate by parts in the appropriate spaces, a natural definition of weak solution of problem (P_λ^s) is the following.

DEFINITION 2.2.1. *We say that* $0<u\in H_{\Sigma_{\mathcal{D}}}^s(\Omega)$ *is a solution of problem* (P_λ^s) *if*

$$\int_\Omega (-\Delta)^{s/2}u\,(-\Delta)^{s/2}\psi dx=\int_\Omega(\lambda u^q+u^r)\,\psi dx,\quad \text{for all } \psi\in H_{\Sigma_{\mathcal{D}}}^s(\Omega).$$

Following the previous definition, we can associate to problem (P_λ^s) the next energy functional,

$$I_\lambda(u)=\frac{1}{2}\int_\Omega|(-\Delta)^{s/2}u|^2dx-\frac{\lambda}{q+1}\int_\Omega|u|^{q+1}dx-\frac{1}{r+1}\int_\Omega|u|^{r+1}dx,\quad u\in H_{\Sigma_{\mathcal{D}}}^s(\Omega), \tag{2.2.1}$$

whose critical points correspond to solutions of (P_λ^s).
Due to the nonlocal nature of the fractional operator $(-\Delta)^s$ some difficulties arise when one tries to obtain explicit expressions involving the action of the fractional Laplacian on a given function. Following the scheme of Chapter 1, we use the extension technique in order to overcome these difficulties. Using the arguments in Section 1.2 we can reformulate problem

(P_λ^s) in terms of the extension problem as follows:

$$(P_\lambda^*) \qquad \left\{ \begin{array}{rl} -\mathrm{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(U) = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ U > 0 & \text{on } \Omega\times\{y=0\} \\ \dfrac{\partial U}{\partial\nu^s} = \lambda U^q + U^r & \text{on } \Omega\times\{y=0\}, \end{array}\right.$$

where

$$B(U) = \chi_{\Sigma_{\mathcal{D}}^*}\cdot U + \chi_{\Sigma_{\mathcal{N}}^*}\cdot\frac{\partial U}{\partial\nu},$$

being ν the outwards normal vector to $\partial_L\mathscr{C}_\Omega$.

Definition 2.2.2. *An* energy solution *to problem* (P_λ^*) *is a function* $U\in\mathcal{X}_{\Sigma_{\mathcal{D}}}^s(\mathscr{C}_\Omega)$, *with* $U>0$ *on* $\Omega\times\{y=0\}$, *such that*

$$\kappa_s\int_{\mathscr{C}_\Omega} y^{1-2s}\langle\nabla U,\nabla\varphi\rangle\, dxdy = \int_\Omega(\lambda U^q(x,0)+U^r(x,0))\,\varphi(x,0)dx,\ \forall\,\varphi\in\mathcal{X}_{\Sigma_{\mathcal{D}}}^s(\mathscr{C}_\Omega) \tag{2.2.2}$$

Given $U\in\mathcal{X}_{\Sigma_{\mathcal{D}}}^s(\mathscr{C}_\Omega)$ a solution of problem (P_λ^*) the function $u(x)=Tr[U(x,y)]=U(x,0)$ belongs to $H_{\Sigma_{\mathcal{D}}}^s(\Omega)$ and solves problem (P_λ^s). Moreover, also the vice versa is true: given a solution $u\in H_{\Sigma_{\mathcal{D}}}^s(\Omega)$, its s-extension $U=E_s[u]\in\mathcal{X}_{\Sigma_{\mathcal{D}}}^s(\mathscr{C}_\Omega)$ is a solution of (P_λ^*). Hence, both formulations are equivalent and the extension operator E_s allows us to switch between them.

2.3. Moving planes and monotonicity

In this section we establish a monotonicity result for bounded solutions to $(-\Delta)^s u = u^r$ in $\mathbb{R}_+^N\equiv\mathbb{R}^{N-1}\times\mathbb{R}_+$ satisfying the boundary conditions:

- $u=0$ on $\Sigma_{\mathcal{D}}(\tau)$ given by
$$\Sigma_{\mathcal{D}}(\tau)=\{(x_1,\cdots,x_N)\in\mathbb{R}^N : x_N=0, x_1\le\tau\},$$
for some $\tau\in\mathbb{R}$.
- $\frac{\partial u}{\partial x_N}=0$ on $\Sigma_{\mathcal{N}}(\tau)$ given by
$$\Sigma_{\mathcal{N}}(\tau)=\{(x_1,\cdots,x_N)\in\mathbb{R}^N : x_N=0, x_1>\tau\},$$
for some $\tau\in\mathbb{R}$.

The principal result proven in this section is the following.

Theorem 2.3.1. *Assume that* $1<r<\frac{N+2s}{N-2s}$, $N>2s$, *and* $\tau\in\mathbb{R}$. *Let* $u\in H_{loc}^s(\mathbb{R}_+^N)\cap\mathcal{C}^0(\overline{\mathbb{R}_+^N})$ *be a weak solution of*

$$\left\{\begin{array}{rl} (-\Delta)^s u = u^r, \quad u>0, & \text{in } \mathbb{R}_+^N, \\ u=0 & \text{on } \Sigma_{\mathcal{D}}(\tau), \\ \frac{\partial u}{\partial x_N}=0 & \text{on } \Sigma_{\mathcal{N}}(\tau). \end{array}\right. \tag{2.3.1}$$

Then, u *is nondecreasing with respect to the* x_1*-direction.*

Remark 2.3.1. *We make the proof assuming* $\tau=0$. *For* $\tau\neq0$ *the proof is analogous through a translation with respect to the variable* x_1.

The proof of Theorem 2.3.1 is based on the method of moving planes introduced by Alexandrov and first exploited in the context of Partial Differential Equations by [**78**] (see also [**59**]).

Let us introduce some notation in order to apply the moving planes method: we denote by $\mathbb{R}^{N+1}_{++} \equiv \mathbb{R}^N_+ \times \mathbb{R}_+$, i.e. the set of points $X = (x, y)$ with $x = (x_1, \ldots, x_N)$ and $x_N, y > 0$. For a fixed $\rho \in \mathbb{R}$, we define the sets

$$\Upsilon_\rho = \{x \in \mathbb{R}^N_+ :\ x_1 < \rho\}, \quad \Upsilon^*_\rho = \Upsilon_\rho \times \mathbb{R}_+,$$

$$T_\rho = \{X \in \overline{\mathbb{R}^{N+1}_{++}} :\ x_1 = \rho\}.$$

For any $X \in \mathbb{R}^{N+1}_{++}$ the reflection with respect to the hyperplane T_ρ is denoted by

$$X^\rho = (x^\rho, y) = X + 2(\rho - x_1)e_1 = (2\rho - x_1, x_2, \ldots, x_N, y).$$

Let us define the point $O_\rho = (2\rho, 0, \ldots, 0, 0) \in \mathbb{R}^{N+1}$, whose reflection is the origin, and $o_\rho = (2\rho, 0, \ldots, 0) \in \mathbb{R}^N$. We also recall that the Kelvin transform of a nontrivial point $x \in \mathbb{R}^N$ is given by $\mathcal{K}(x) = \frac{x}{|x|^2}$. It is easy to see that $\mathcal{K}(\mathbb{R}^N_+) = \mathbb{R}^N_+$ and $\mathcal{K}\left(\Upsilon^*_\rho\right) = (\mathbb{R}^{N+1}_{++}) \cap B_{\frac{1}{-4\rho}}(O_{\frac{1}{4\rho}})$ for any $\rho < 0$. Next, we follow the approach of [**24**] based on the fractional Kelvin transform,

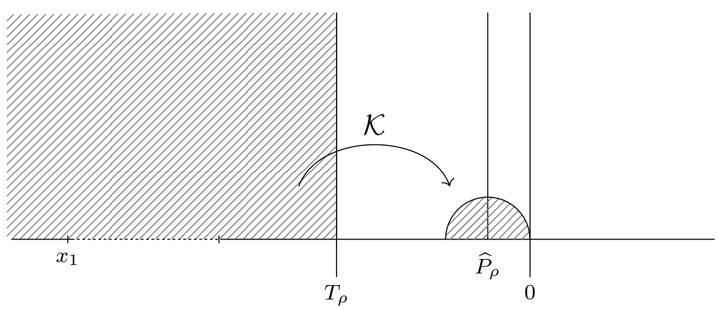

FIGURE 1. The Kelvin Transform acting on the set Υ^*_ρ, with $\rho < 0$.

$\mathcal{K}_s(u)$, acting on functions defined in a subset of $\mathbb{R}^N$, in the following way:

$$\mathcal{K}_s(u) = \frac{1}{|x|^{N-2s}} u\left(\mathcal{K}(x)\right) = \frac{1}{|x|^{N-2s}} u\left(\frac{x}{|x|^2}\right).$$

As it is proven in [**24**], if $(-\Delta)^s u = f(u)$, then the action of the fractional laplacian acting on the fractional Kelvin transform of u is given by

$$(-\Delta)^s \mathcal{K}_s(u) = \frac{1}{|x|^{N+2s}} f\left(u(\mathcal{K}(x))\right).$$

Let $u(x)$ be a solution of problem (2.3.1) and define $f(t) = t^r$ and $g(t) = \dfrac{f(t)}{t^{\frac{N+2s}{N-2s}}}$. Then, the Kelvin transform $v = \mathcal{K}_s(u)$ satisfies the same mixed BVP as u, namely

$$\begin{cases} (-\Delta)^s v = g(|x|^{N-2s} v) v^{\frac{N+2s}{N-2s}}, \quad v > 0, & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{on } \Sigma_{\mathcal{D}}(0), \\ \dfrac{\partial v}{\partial x_N} = 0 & \text{on } \Sigma_{\mathcal{N}}(0), \end{cases}$$

since on $x_N = 0$, we have

$$\frac{\partial v}{\partial x_N}(x) = (2s-N)\frac{x_N}{|x|^{N+2(1-s)}}u\left(\mathcal{K}(x)\right) + \frac{1}{|x|^{N-2s}}\frac{\partial u}{\partial x_N}\left(\mathcal{K}(x)\right) = 0.$$

Moreover, v is a continuous and positive function in $\mathbb{R}^N\backslash\{0\}$, with a possible singularity at the origin and decays at infinity as $\frac{1}{|x|^{N-2s}}u(0)$, thus $v \in L^{2_s^*} \cap L^\infty(\mathbb{R}^N_+\backslash B_r(0))$ for any $r > 0$. Finally, we consider $V = E_s[v]$ the extension function of the Kelvin transform $v = \mathcal{K}_s(u)$ and the corresponding extension problem,

$$\left\{\begin{aligned} -\mathrm{div}(y^{1-2s}\nabla V) &= 0 && \text{in } \mathbb{R}^{N+1}_{++} \subset \mathbb{R}^{N+1}_+,\\ B(V) &= 0 && \text{on } (\Sigma_{\mathcal{D}}(0) \cup \Sigma_{\mathcal{N}}(0)) \times \mathbb{R}_+,\\ \frac{\partial U}{\partial \nu^s} &= g(|x|^{N-2s}v)v^{\frac{N+2s}{N-2s}} && \text{on } \Omega \times \{y=0\}.\end{aligned}\right. \tag{2.3.2}$$

Observe that, since $v \in L^{2_s^*}(\mathbb{R}^N_+\backslash B_r(0))$ for any $r > 0$ and the extension operator E_s is an isometry, by [**48**], the extension function $V \in L^{\overline{2}^*}(\Upsilon^*_\rho, y^{1-2s}dX)$ for any $\rho < 0$, where $\overline{2}^* = \frac{2(N+1)}{N-1}$ denotes to the Sobolev conjugate exponent in dimension $N+1$.
The following lemma, which extends to our fractional framework [**37**, Lemma 2.1], provides us with a key-point inequality in order to obtain monotonicity in the x_1-direction for the function V defined in (2.3.2).

Here we use the notation $V_\rho(X) = V(X^\rho)$ and $v_\rho(x) = v(x^\rho)$ for the reflected functions that are singular at the point O_ρ and o_ρ respectively. Moreover we denote by $\mathcal{A}_\rho = \{x \in \Upsilon_\rho\backslash O_\rho : v \geq v_\rho\}$.

Lemma 2.3.1. *Assume that $u \in H^s_{loc}(\mathbb{R}^N_+) \cap \mathcal{C}^0(\overline{\mathbb{R}^N_+})$ is a weak solution of* (2.3.1) *and let $v = \mathcal{K}_s(u)$. Then, for any $\rho < 0$, $(v - v_\rho)^+ \in H^s_{\Sigma_{\mathcal{D}}}(\Upsilon_\rho) \cap L^\infty(\Upsilon_\rho)$. Moreover, there exists $C_\rho > 0$, increasing with respect to ρ, such that*

$$\int_{\Upsilon^*_\rho} y^{1-2s}|\nabla(V - V_\rho)^+|^2 dxdy \leq C_\rho\left(\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}}\int_{\Upsilon^*_\rho} y^{1-2s}|\nabla(V - V_\rho)^+|^2 dxdy. \tag{2.3.3}$$

Proof. Since for a given $\rho < 0$ there exists $r > 0$ such that $\Upsilon_\rho \subset \mathbb{R}^N_+\backslash B_r(0)$, the functions v and $(v - v_\rho)^+ \leq v$ belong to $L^{2_s^*}(\Upsilon_\rho) \cap L^\infty(\Upsilon_\rho)$ and the function $\frac{1}{|x|^{2N}}$ is integrable in Υ_ρ. The assertion $(v - v_\rho)^+ \in H^s_{\Sigma_{\mathcal{D}}}(\Upsilon_\rho)$ follows from (2.3.3) taking in mind that the extension operator E_s is an isometry. To prove estimate (2.3.3) we test conveniently the equations

$$(-\Delta)^s v = g(|x|^{N-2s}v)v^{\frac{N+2s}{N-2s}}, \quad (-\Delta)^s v_\rho = g(|x^\rho|^{N-2s}v_\rho)v_\rho^{\frac{N+2s}{N-2s}},$$

in the set $\Upsilon_\rho\backslash O_\rho$. At this point, we make full use of the extension technique, so that we consider the extension functions $V = E_s[v]$ and $V_\rho = E_s[v_\rho] = V(X^\rho)$ and we set the nonnegative function $\varphi = \varphi_\varepsilon = \eta^2_\varepsilon(V - V_\rho)^+$ as a test function in the corresponding extended problem for a convenient function η_ε. More precisely, for $\varepsilon > 0$ small enough we take $\eta_\varepsilon \in$

$\mathcal{C}_0^1(\mathbb{R}^{N+1})$ with $0 \leq \eta_\varepsilon \leq 1$ and such that:

$$\begin{cases} \eta_\varepsilon \equiv 1 & \text{for } 2\varepsilon \leq |X-O_\rho| \leq \dfrac{1}{\varepsilon} \\ \eta_\varepsilon \equiv 0 & \text{for } |X-O_\rho| \leq \varepsilon \quad \text{or} \quad \dfrac{2}{\varepsilon} \leq |X-O_\rho|, \\ |\nabla \eta_\varepsilon| \leq \dfrac{c}{\varepsilon} & \text{for } \varepsilon < |X-O_\rho| < 2\varepsilon \\ |\nabla \eta_\varepsilon| \leq c\varepsilon & \text{for } \dfrac{1}{\varepsilon} < |X-O_\rho| < \dfrac{2}{\varepsilon}. \end{cases}$$

Observe that in the set Υ_ρ^* the function $(V-V_\rho)^+$ vanishes where the Dirichlet condition holds for V but also where the Dirichlet condition holds for the reflected function and, therefore, it is allowed to take $\varphi = \eta_\varepsilon^2(V-V_\rho)^+$ as a test function in the corresponding extended problem. Thus, using the definition of weak solution for the extended problem satisfied for V and V_ρ

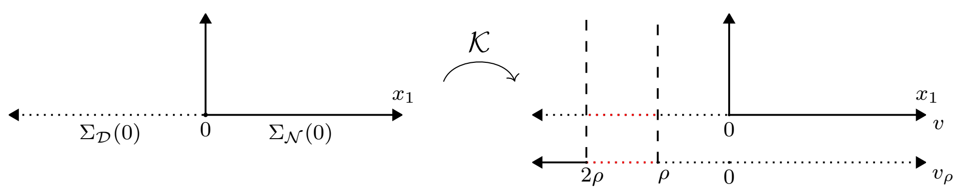

FIGURE 2. The Kelvin transform centered at 0 acting on $\Sigma_{\mathcal{D}}(0)$ (doted line) and $\Sigma_{\mathcal{N}}(0)$ for the functions v and v_ρ.

respectively and subtracting those expressions, we obtain

$$\kappa_s \int_{\Upsilon_\rho^*} y^{1-2s} \nabla(V-V_\rho)\nabla\varphi \, dxdy = \int_{\Upsilon_\rho} \left(g(|x|^{N-2s}v)v^{\frac{N+2s}{N-2s}} - g(|x^\rho|^{N-2s}v_\rho)v_\rho^{\frac{N+2s}{N-2s}} \right) \varphi(x,0)dx.$$

On the other hand,

$$\begin{aligned} \kappa_s \int\limits_{\Upsilon_\rho^* \cap [2\varepsilon \leq |X-O_\rho| \leq \frac{1}{\varepsilon}]} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy &\leq \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}|\nabla(\eta_\varepsilon(V-V_\rho)^+)|^2 dxdy \\ &= \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}\nabla(V-V_\rho)\nabla\varphi \, dxdy + \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}[(V-V_\rho)^+]^2|\nabla\eta_\varepsilon|^2 dxdy \\ &= \kappa_s \int_{\Upsilon_\rho^*} y^{1-2s}\nabla(V-V_\rho)\nabla\varphi \, dxdy + I_\varepsilon \\ &= \int_{\Upsilon_\rho} \left(g(|x|^{N-2s}v)v^{\frac{N+2s}{N-2s}} - g(|x^\rho|^{N-2s}v_\rho)v_\rho^{\frac{N+2s}{N-2s}} \right) \varphi(x,0)dx + I_\varepsilon. \end{aligned}$$

Since g is a nonincreasing function, $|x| \geq |x^\rho|$ in Υ_ρ and $v \geq v_\rho$ in the set where $\varphi(\cdot, 0) \neq 0$, it follows that $-g(|x^\rho|^{N-2s}v_\rho) \leq -g(|x|^{N-2s}v)$ and, therefore,

(2.3.4)
$$\begin{aligned}\kappa_s \int\limits_{\Upsilon_\rho^* \cap [2\varepsilon \leq |X-O_\rho| \leq \frac{1}{\varepsilon}]} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy &\leq \int_{\Upsilon_\rho} g(|x|^{N-2s}v)\left(v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}}\right)\varphi(x,0)dx + I_\varepsilon \\ &\leq \int_{\mathcal{A}_\rho} g(|x|^{N-2s}v)\left(v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}}\right)\varphi(x,0)dx + I_\varepsilon.\end{aligned}$$

Now, if $0 \leq v_\rho \leq v$ from the Mean Value Theorem, we find

$$v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}} \leq \frac{N+2s}{N-2s} v^{\frac{4s}{N-2s}}(v - v_\rho).$$

On the other hand, since $f(t) = t^r$ with $1 < r < \frac{N+2s}{N-2s}$, it follows that

$$g(t)t^{\frac{4s}{N-2s}} = \frac{f(t)}{t^{\frac{N+2s}{N-2s}}} t^{\frac{4s}{N-2s}} = \frac{f(t)}{t} = t^{r-1},$$

and $g(t)t^{\frac{4s}{N-2s}}$ is bounded in any interval $(0, t_0)$. Moreover, since $|x|^{N-2s}v(x) = u\left(\frac{x}{|x|^2}\right)$ is bounded from above for $x \in \Upsilon_\rho$ and $\rho < 0$, we conclude

$$\begin{aligned}g(|x|^{N-2s}v)\left(v^{\frac{N+2s}{N-2s}} - v_\rho^{\frac{N+2s}{N-2s}}\right) &\leq \frac{N+2s}{N-2s} g(|x|^{N-2s}v)v^{\frac{4s}{N-2s}}(v - v_\rho) \\ &\leq \frac{N+2s}{N-2s} \frac{g(|x|^{N-2s}v)(|x|^{N-2s}v)^{\frac{4s}{N-2s}}}{|x|^{4s}}(v - v_\rho) \\ &\leq \widetilde{C}_\rho \frac{1}{|x|^{4s}}(v - v_\rho),\end{aligned}$$

for a positive constant $\widetilde{C}_\rho$ increasing in ρ. Then, inequality (2.3.4) takes the form

$$\begin{aligned}\kappa_s \int\limits_{\Upsilon_\rho^* \cap [2\varepsilon \leq |X-O_\rho| \leq \frac{1}{\varepsilon}]} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy &\leq \widetilde{C}_\rho \int_{\mathcal{A}_\rho} \frac{1}{|x|^{4s}}(v - v_\rho)\varphi(x,0)dx + I_\varepsilon \\ &\leq \widetilde{C}_\rho \int_{\mathcal{A}_\rho} \frac{1}{|x|^{4s}}\eta_\varepsilon^2(x,0)[(v - v_\rho)^+]^2 dx + I_\varepsilon \\ &\leq \widetilde{C}_\rho \int_{\mathcal{A}_\rho} \frac{1}{|x|^{4s}}[(v - v_\rho)^+]^2 dx + I_\varepsilon.\end{aligned}$$

Using Hölder's inequality with $p = \frac{N}{2s}$ and $q = \frac{2_s^*}{2}$ we conclude

$$\kappa_s \int\limits_{\Upsilon_\rho^* \cap [2\varepsilon \leq |X-O_\rho| \leq \frac{1}{\varepsilon}]} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy \leq \widetilde{C}_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}} \left(\int_{\Upsilon_\rho} [(v - v_\rho)^+]^{2_s^*} dx\right)^{\frac{2}{2_s^*}} + I_\varepsilon.$$

Next, we focus on the term $I_\varepsilon = \displaystyle\int_{\Upsilon_\rho^*} y^{1-2s}[(V-V_\rho)^+]^2|\nabla\eta_\varepsilon|^2 dxdy$. Define the set

$$\mathcal{W}_\varepsilon = \left\{X \in \Upsilon_\rho^* : \ \varepsilon < |X - O_\rho| < 2\varepsilon \text{ or } \frac{1}{\varepsilon} < |X - O_\rho| < \frac{2}{\varepsilon}\right\},$$

so that $supp(|\nabla\eta_\epsilon|^2) \subseteq \overline{\mathcal{W}_\varepsilon}$. Since $\left||\nabla\eta_\varepsilon|^{N+1}\chi_{\mathcal{W}_\varepsilon}\right| \leq c(\frac{1}{\varepsilon^{N+1}}\varepsilon^{N+1} + \varepsilon^{N+1}\frac{1}{\varepsilon^{N+1}}) = c'$ and $(V-V_\rho)^+ \in L^{\overline{2}^*}(\Upsilon_\rho^*, y^{1-2s}dxdy)$, applying Hölder's inequality with $p = \frac{N+1}{2}$ and $q = \frac{\overline{2}^*}{2}$, we find

$$\begin{aligned}
I_\varepsilon &\leq \left(\int_{\mathcal{W}_\varepsilon} y^{1-2s}[(V-V_\rho)^+]^{\overline{2}^*}dxdy\right)^{\frac{2}{\overline{2}^*}} \left(\int_{\mathcal{W}_\varepsilon} y^{1-2s}|\nabla\eta_\varepsilon|^{N+1}dxdy\right)^{\frac{2}{N+1}} \\
&\leq C\left(\int_{\mathcal{W}_\varepsilon} y^{1-2s}[(V-V_\rho)^+]^{\overline{2}^*}dxdy\right)^{\frac{2}{\overline{2}^*}} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Therefore, applying the trace inequality (1.2.9), we conclude

$$\begin{aligned}
\int_{\Upsilon_\rho^*} y^{1-2s}|\nabla(V-V_\rho)^+|^2 dxdy &\leq \kappa_s^{-1}\widetilde{C}_\rho\left(\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}}\left(\int_{\Upsilon_\rho}[(v-v_\rho)^+]^{2_s^*}dx\right)^{\frac{2}{2_s^*}} \\
&\leq C_\rho\left(\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}}\int_{\Upsilon_\rho^*} y^{1-2s}|\nabla(V-V_\rho)^+|^2dxdy,
\end{aligned}$$

for a positive constant C_ρ increasing with respect to ρ. □

Proof of Theorem 2.3.1. The proof follows the lines of [**37**, Proposition 2.1]. First, we establish a starting plane that delimits an hyperspace in which the monotonicity in the x_1-direction holds. Next we extend such a region progressively until we reach the half-space, and in a second step, to the whole space having a special care to the singularity of the Kelvin transform at the origin. Since

$$\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx \leq \int_{\Upsilon_\rho}\frac{1}{|x|^{2N}}dx \to 0, \text{ as } \rho \to -\infty,$$

then there exists $-\infty < \rho_0 < 0$ such that

$$C_\rho\left(\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx\right)^{\frac{2s}{N}} < 1, \quad \text{for all} \quad \rho \in (-\infty, \rho_0).$$

From (2.3.3) we deduce that $(V-V_\rho)^+ \equiv 0$ in Υ_ρ^*, and therefore $V \leq V_\rho$ in Υ_ρ^* for all $\rho \in (-\infty, \rho_0)$. Consequently $v \leq v_\rho$ in Υ_ρ for any $\rho \in (-\infty, \rho_0)$.

Assume now that $\rho_0 < 0$ is maximal. By the Maximum Principle $v < v_{\rho_0}$ in Υ_{ρ_0}. Then $\chi_{\mathcal{A}_\rho}\cdot\frac{1}{|x|^{2N}} \to 0$ point-wisely as $\rho \to \rho_0$ in $\mathbb{R}^N_+\backslash\{T_{\rho_0}\cup\{O_{\rho_0}\}\}$.

Thus, if $\rho < \rho_0 + \delta < 0$ then $\chi_{\mathcal{A}_\rho}\cdot\frac{1}{|x|^{2N}} \leq \chi_{\Upsilon_{\rho_0+\delta}}\cdot\frac{1}{|x|^{2N}} \in L^1(\mathbb{R}^N_+)$ so that applying the Dominated Convergence Theorem

$$\int_{\mathcal{A}_\rho}\frac{1}{|x|^{2N}}dx \to 0, \quad \text{as } \rho \to \rho_0,$$

and we conclude

$$C_\rho \left(\int_{\mathcal{A}_\rho} \frac{1}{|x|^{2N}} dx \right)^{\frac{2s}{N}} < 1, \ \forall \rho \in (\rho_0, \rho_0 + \delta),$$

for some $\delta > 0$ small enough. Therefore $(V - V_\rho)^+ \equiv 0$ in Υ^*_ρ for $\rho \in (\rho_0, \rho_0 + \delta)$ in contradiction with the maximality of ρ_0. As a consequence $V < V_\rho$ in Υ^*_ρ provided $\rho < 0$ and by continuity $V \le V_0$ in Υ^*_0, so that $v \le v_0$ in Υ_0. Noticing that $|x| = |x^\rho|$ for $\rho = 0$ we conclude $u \le u_0$ in Υ_0.
The above argument works for the Kelvin transform centered at a point $P = P_\mu = (\mu, 0, \dots, 0) \in \mathbb{R}^N_+$, namely, $v^\mu(x) = \frac{1}{|x|^{N-2s}} u(P_\mu + \frac{x}{|x|^2})$ with $\mu \le 0$ (see Figure 3). This centered fractional

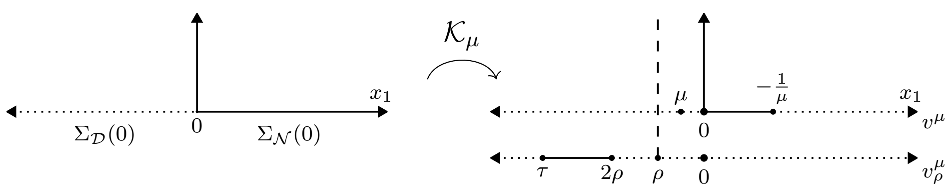

FIGURE 3. The Kelvin transform centered at $\mu \le 0$ acting on $\Sigma_{\mathcal{D}}(0)$ (doted line) and $\Sigma_{\mathcal{N}}(0)$ for the functions v^μ and v^μ_ρ. The set $\Sigma_{\mathcal{N}}(0)$ is transformed into those $x \in \mathbb{R}^N_+$ such that $0 < x_1 < -\frac{1}{\mu}$, so v^μ_ρ satisfies a Neumann condition on $\tau < x_1 < 2\rho$ with $\tau = 2\rho + \frac{1}{\mu}$.

Kelvin transform v^μ satisfies a Dirichlet condition in the part of the boundary with $x_N = 0$ and $x_1 < 0$ so we can prove as before that for any $\rho < 0$ the inequality $v^\mu \le v^\mu_\rho$ holds in Υ_ρ. Since $\rho < 0$ is arbitrary it follows that $v^\mu \le v^\mu_0$ in Υ_0. Thus $u \le u_\mu$ in Υ_μ for $\mu \le 0$, so u is nondecreasing in the x_1-direction provided $x_1 < 0$.

Now we extend progressively the region in which the monotonicity holds reaching Υ_μ for $\mu > 0$. First, observe that we can not continue as before due to the singularity of the Kelvin transform at the origin: we cannot take a moving plane starting at $\rho = -\infty$ since for ρ large there are points where the Neumann boundary condition holds (and the solution is positive) which are reflected to the Dirichlet part of the boundary. In terms of the test functions, for ρ large enough the function $(V - V_\rho)^+$ is not allowed to be chosen as test function for the problem satisfied by the reflected function V_ρ , since it does not vanish at those points of the boundary where the Dirichlet condition for V_ρ holds.

Nevertheless, an inequality similar to (2.3.3) holds for $(v^\mu - v^\mu_\rho)^+$ if ρ is close to 0 so that we extend the inequality $v^\mu(x) < v^\mu_\rho(x) = v^\mu(x_\rho)$ for every $\rho < 0$ fixed, moving μ from $\mu = 0$ where the strict inequality is true up to $\mu = \frac{-1}{2\rho}$.
If $\mu \ge 0$, the fractional Kelvin transform centered at the point P_μ (denoted by $v^\mu(x)$) satisfies a Dirichlet boundary condition at points $x \in \mathbb{R}^N_+$ with $x_N = 0$ and $\frac{-1}{\mu} < x_1 < 0$ ($x_1 < 0$ if $\mu = 0$ as in the previous step) and a Neumann condition on the remaining part of the boundary. Then, if $-\frac{1}{2\mu} < \rho < 0$ it follows that V^μ, and hence $(V^\mu - V^\mu_\rho)^+$, vanishes where the Dirichlet condition holds for V^μ and also where the Dirichlet condition holds for the reflected function V^μ_ρ (therefore φ_ε is an allowed test function). Thus, proceeding exactly as

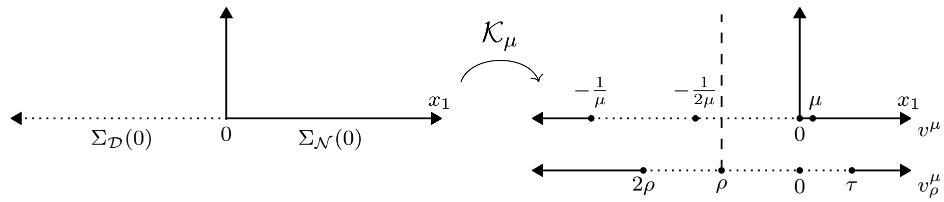

FIGURE 4. The Kelvin transform centered at $\mu \geq 0$ acting on $\Sigma_{\mathcal{D}}(0)$ (doted line) and $\Sigma_{\mathcal{N}}(0)$ for the functions v^μ and v^μ_ρ. The set $\Sigma_{\mathcal{D}}(0)$ is transformed into the $x \in \mathbb{R}^N_+$ such that $x_N = 0$ and $-\frac{1}{\mu} < x_1 < 0$, so the reflected function v^μ_ρ satisfies a Dirichlet condition on $2\rho < x_1 < \tau$ with $\tau = 2\rho + \frac{1}{\mu}$. It follows that for $x \in \Upsilon_\rho$ the function v^μ vanish where the Dirichlet condition holds for v^μ_ρ.

in the case $\mu = 0$, we obtain

$$\int_{\Upsilon^*_\rho} y^{1-2s}|\nabla(V^\mu - V^\mu_\rho)^+|^2 dxdy \leq C_\rho \left(\int_{\mathcal{A}^\mu_\rho} \frac{1}{|x|^{2N}} dx\right)^{\frac{2s}{N}} \int_{\Upsilon^*_\rho} y^{1-2s}|\nabla(V^\mu - V^\mu_\rho)^+|^2 dxdy,$$

where C_ρ is increasing in ρ and $\mathcal{A}^\mu_\rho = \{x \in \Upsilon_\rho \backslash O_\rho : v^\mu \geq v^\mu_\rho\}$.
If we now fix $\rho < 0$ the previous estimate holds for any $\mu \in (0, -\frac{1}{2\rho})$ and, since $\frac{1}{|x|^{2N}} \in L^1(\Upsilon_\rho)$, applying the Dominated Convergence Theorem we conclude $\chi_{\mathcal{A}^\mu_\rho} \cdot \frac{1}{|x|^{2N}} \to 0$ as $\mu \to 0$ in $\mathbb{R}^N \backslash \{T_\rho \cup P_\rho\}$, we recall that $P_\rho = (2\rho, 0, \ldots, 0)$ is the reflected point of the origin, which is the singular point of every transform V^μ. As a consequence

$$C_\rho \left(\int_{\mathcal{A}^\mu_\rho} \frac{1}{|x|^{2N}} dx\right)^{\frac{2s}{N}} < 1,$$

for some $\rho_0 \in (\frac{-1}{2\mu}, 0)$ and the monotonicity follows. Finally, suppose that $\mu_0 < -\frac{1}{2\rho_0}$ is maximal such that $v^\mu \leq v^\mu_\rho$ in Υ_ρ for all $0 < \mu < \mu_0$. Then, by the maximum principle, $v^\mu < v^\mu_\rho$ and hence $\mathcal{A}^\mu_\rho \to \emptyset$ as $\mu \to \mu_0$. Thus, there exists $\epsilon > 0$ such that

$$C_\rho \left(\int_{\mathcal{A}^\mu_\rho} \frac{1}{|x|^{2N}} dx\right)^{\frac{2s}{N}} < 1 \text{ for } \mu \in (\mu_0, \mu_0 + \epsilon).$$

We conclude that $v^\mu \leq v^\mu_\rho$ for $\mu > \mu_0$ and close to μ_0 in contradiction with the maximality of μ_0.
In summary, for every $\rho < 0$ and $\mu \leq -\frac{1}{2\rho}$ we have $v^\mu \leq v^\mu_\rho$ in Υ_ρ or, equivalently, fixed $\mu > 0$ the inequality holds for every $-\frac{1}{2\mu} < \rho < 0$. Letting $\rho \to 0$ we get $v^\mu \leq v^\mu_0$ in Υ_0 i.e. $v^\mu(x_1, x') \leq v^\mu(-x_1, x')$ for all x with $x_1 < 0$, so that $u \leq u_\mu$ in Υ_μ with $\mu > 0$. Since $\mu > 0$ is arbitrary we get that u is nondecreasing in the x_1-direction in whole $\mathbb{R}^N_+$. □

REMARK 2.3.2. *Let us observe that the method described in the above Theorem in the x_1 direction may be applied to any other direction $x_2, \ldots, x_{N-1}$, centered at any point P of the form $P = (0, P_2, \ldots, P_{N-1}, 0)$, with an hyperplane orthogonal to both to the e_1 and e_n*

directions. Thus, due to the arbitrary of the point P, we can deduce that u does not depend to the $x_2, \ldots, x_{N-1}$ variables.

2.4. A priori bounds in $L^\infty(\Omega)$

In this section we prove Theorem 2.1.2 exploiting the blow-up method by Guidas-Spruck (see [**60**]). To this aim we will make use of the estimates proved in Theorem 1.1.1 that guarantee the compactness needed in order to accomplish this limit step. Then, with the same ideas, we prove Theorem 1.3 using the uniform estimates proved in Corollary 1.1.1 for the moving boundary conditions (as in hypotheses (B_1)-(B_3)).

Proof of Theorem 2.1.2. We argue by contradiction: set $\Lambda > 0$ given by Theorem 2.1.1 and assume that there exists sequences $\{\lambda_k\} \subset [0, \Lambda]$, $\{u_k\}$ of solutions to problems (P_{λ_k}) and $\{p_k\} \subset \overline{\Omega}$ of points verifying

$$M_k = \sup_{x \in \overline{\Omega}} \; u_k(x) = u_k(p_k) \to +\infty, \quad \text{as } k \to \infty.$$

Let us set $\mu_k = M_k^{-\frac{r-1}{2s}}$ and define the functions $v_k(y) = \frac{1}{M_k} u(p_k + \mu_k y)$. Note that $v_k(y)$ is defined in $\Omega_k = \frac{1}{\mu_k}(\Omega - p_k)$ as well as $v_k(0) = 1$ and $\|v_k\|_{L^\infty(\Omega_k)} \leq 1$ for all $k \geq 0$. Moreover, the rescaled function v_k satisfies the problem

$$\left\{\begin{array}{lll} (-\Delta)^s v_k = \lambda_k M_k^{q-r} v_k^q + v_k^r & v_k > 0, & \text{in } \Omega_k = \frac{1}{\mu_k}(\Omega - p_k), \\ \qquad v_k = 0 & & \text{on } \Sigma_{\mathcal{D}}^k, \\ \qquad \dfrac{\partial v_k}{\partial \nu} = 0 & & \text{on } \Sigma_{\mathcal{N}}^k, \end{array}\right.$$

where $\Sigma_{\mathcal{D}}^k$ and $\Sigma_{\mathcal{N}}^k$ are the transformed boundary manifolds.

Now we study the limit problem obtained as $k \to \infty$. To carry out this step we need some compactness properties for the sequence $\{v_k\}$ in order to guarantee the convergence in some sense. Because of Theorem 1.1.1 the sequence $\{v_k\}$ is uniformly bounded in $\mathcal{C}^\gamma(\Omega_k)$ for some $\gamma \in \left(0, \frac{1}{2}\right)$. Then, by the Ascoli-Arzelá Theorem, there exists a subsequence $\{v_k\}$ locally uniformly convergent in $\mathbb{R}_+^N$ to a function $v \in C^\gamma(\overline{\mathbb{R}_+^N})$ for some $0 < \gamma < \frac{1}{2}$. Moreover $\|v\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and $v(0) = 1$.
On the other hand, the problem satisfied by the limit function v depends on the position of the point $p = \lim\limits_{k \to \infty} p_k$. Let us set

$$d_k^{\mathcal{D}} = dist(p_k, \Sigma_{\mathcal{D}}^k) \quad \text{and} \quad d_k^{\mathcal{N}} = dist(p_k, \Sigma_{\mathcal{N}}^k).$$

and define $d_k^\Omega = \min\{d_k^{\mathcal{D}}, d_k^{\mathcal{N}}\}$. We distinguish several cases according to the behavior of the sequences $\frac{d_k^i}{\mu_k}$ with $i = \Omega, \mathcal{D}, \mathcal{N}$.

1. **Interior case:** $\left\{\frac{d_k^\Omega}{\mu_k}\right\} \to +\infty$.

Since $B_{d_k^\Omega/\mu_k}(0) \subset \Omega_k$ (see Figure 5) we have that $\Omega_k \to \mathbb{R}^N$ and the limit function v is a positive bounded solution of

$$(-\Delta)^s v = v^r \quad \text{in } \mathbb{R}^N,$$

Then, by [**33**, Theorem 1] (see also [**24**, Theorem 3.1])we conclude $v \equiv 0$, in contradiction with $v(0) = 1$.

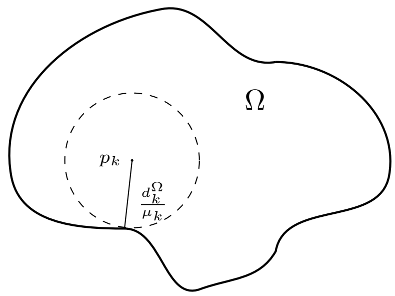

FIGURE 5. The relevant geometry after dilation of variables lies in a neighbourhood of p_k such as the one of the picture.

2. **Boundary Cases:** $\left\{\frac{d_k^{\Omega}}{\mu_k}\right\} \to d^{\Omega} \in \mathbb{R}^+$.

In this situation we have several possibilities:

2.1 Dirichlet Case: $\left\{\frac{d_k^{\mathcal{D}}}{\mu_k}\right\} \to d^{\mathcal{D}} \in \mathbb{R}^+$ and $\left\{\frac{d_k^{\mathcal{N}}}{\mu_k}\right\} \to +\infty$.
Now, as $\Sigma_{\mathcal{D}}$ is a $(N-1)$-dimensional smooth manifold, we have that, up to a rotation

$$\Omega_k \to \Omega_{d^{\mathcal{D}}} \equiv \{x \in \mathbb{R}^N : x_N > -d^{\mathcal{D}}\},$$

and the limit function v is a positive solution of

$$\begin{cases} (-\Delta)^s v = v^r & \text{in } \Omega_{d^{\mathcal{D}}}\ , \\ \qquad v = 0 & \text{in } \{x_N = -d^{\mathcal{D}}\}, \end{cases}$$

with $\|v\|_{L^{\infty}(\Omega_{d^{\mathcal{D}}})} \leq 1$ and $v(0) = 1$. Thus, if $d^{\mathcal{D}} = 0$ we have a contradiction with the continuity since $v(0) = 1$ while if $d^{\mathcal{D}} > 0$ we have a contradiction with [**24**, Theorem 3.4]

2.2 Neumann case: $\left\{\frac{d_k^{\mathcal{D}}}{\mu_k}\right\} \to +\infty$ and $\left\{\frac{d_k^{\mathcal{N}}}{\mu_k}\right\} \to d^{\mathcal{N}} \in \mathbb{R}^+$.
As before, since $\Sigma_{\mathcal{N}}$ is a $(N-1)$-dimensional smooth manifold, we have that, up to rotation,

$$\Omega_k \to \Omega_{d^{\mathcal{N}}} \equiv \{x \in \mathbb{R}^N : x_N > -d^{\mathcal{N}}\},$$

and the limit function v is a positive solution of

$$\begin{cases} (-\Delta)^s v = v^r & \text{in } \Omega_{d^{\mathcal{N}}}, \\ \quad \frac{\partial v}{\partial x_N} = 0 & \text{in } \{x_N = -d^{\mathcal{N}}\}, \end{cases}$$

with $\|v\|_{L^{\infty}(\Omega_{d^{\mathcal{N}}})} \leq 1$ and $v(0) = 1$. Then, if we define the translated function $w(x) = v(x_1, x_2, \ldots, x_N + d^{\mathcal{N}})$ it follows that

$$\begin{cases} (-\Delta)^s w = w^r & \text{in } \mathbb{R}^N_+, \\ \quad \frac{\partial w}{\partial x_N} = 0 & \text{in } \{x_N = 0\}, \end{cases}$$

with $\|w\|_{L^{\infty}(\mathbb{R}^N_+)} \leq 1$ and $w(0, 0, \ldots, d^{\mathcal{N}}) = 1$. Extending to the whole space by reflection through the hyperplane $\{x_N = 0\}$, thanks to [**24**, Theorem 3.1], it follows that $w \equiv 0$ and we get a contradiction with $w(0, 0, \ldots, d^{\mathcal{N}}) = 1$.

2.3 Interphase Case: $\left\{\frac{d_k^{\mathcal{D}}}{\mu_k}\right\} \to d^{\mathcal{D}} \in \mathbb{R}^+$ and $\left\{\frac{d_k^{\mathcal{N}}}{\mu_k}\right\} \to d^{\mathcal{N}} \in \mathbb{R}^+$.

Let us set $d^\Omega = \min\{d^{\mathcal{D}}, d^{\mathcal{N}}\} \geq 0$ and note that $\Sigma_{\mathcal{D}}^k$, $\Sigma_{\mathcal{N}}^k$ and $\Gamma_k = \Sigma_{\mathcal{D}}^k \cap \overline{\Sigma_{\mathcal{N}}^k}$ are smooth manifolds by hypotheses $(\mathfrak{B})$. Hence, we can assume that, up to a rotation,

$$\Omega_k \to \Omega_{d^\Omega} \equiv \{x \in \mathbb{R}^N : x_N > -d^\Omega\},$$

and the interphase $\Gamma_k \to \{x_1 = \tau\}$ for some finite $\tau \in \mathbb{R}$. Then the limit function v is a positive solution of

$$\begin{cases} (-\Delta)^s v = v^r & \text{in } \Omega_{d^\Omega}, \\ v = 0 & \text{in } \{x_N = -d^\Omega\} \cap \{x_1 \leq \tau\}, \\ \frac{\partial v}{\partial x_N} = 0 & \text{in } \{x_N = -d^\Omega\} \cap \{x_1 > \tau\}, \end{cases}$$

with $\|v\|_{L^\infty(\Omega_{d^\Omega})} \leq 1$ and $v(0) = 1$.

1) If $d^\Omega = 0$ and $\tau \geq 0$ we get a contradiction with the continuity of v, since the maximum is achieved at a point on the Dirichlet boundary where $v \equiv 0$.
2) If $d^\Omega > 0$ and $\tau \geq 0$ we get a contradiction with the monotonicity (Theorem 2.3.1) and the Hopf Lemma at the maximum point. Indeed it is sufficient to have the monotonicity of the solution v with respect to the x_1-direction up to $x_1 = \tau$.
3) If $\tau < 0$, we reach, once again, a contradiction with the monotonicity and the Hopf Lemma at the point of maximum. In this step it is necessary to use the monotonicity of v with respect to the x_1-direction in the whole space.

□

With the same ideas, we can prove the next result concerning the moving boundary conditions.

Proof of Theorem 2.1.3. As we did in Theorem 2.1.2, we argue by contradiction. Assume that there exists a sequence of solutions $\{u_\alpha\}_{\alpha \in I_\varepsilon}$ of solutions to problems $(P_{\alpha,\lambda})$, a sequence of points $\{p_\alpha\} \subset \overline{\Omega}$, $\overline{\alpha} \in I_\varepsilon$ and a sequence of numbers $\mu_\alpha = M_\alpha^{\frac{1-r}{2s}}$ verifying

$$M_\alpha = \sup_{x \in \overline{\Omega}} u_\alpha(x) = u_\alpha(p_\alpha) \to +\infty, \text{ as } \alpha \to \overline{\alpha}\,.$$

We have to distinguish several cases. The interior, Dirichlet and Neumann cases can be proved following the corresponding cases in Theorem 2.1.2. For the interphase case, we need some compactness for the sequence $\{u_\alpha\}$ as $\alpha \to \overline{\alpha}$. Since we are considering sets $\Sigma_{\mathcal{D}}(\alpha)$ with $\alpha \in I_\varepsilon = [\varepsilon, |\partial\Omega|]$ for some $\varepsilon > 0$ and satisfying hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3), by Corollary 1.1.1 the sequence $\{u_\alpha\}$ is uniformly bounded in $\mathcal{C}^\gamma(\overline{\Omega})$ for some $\gamma \in \left(0, \frac{1}{2}\right)$ and so the conclusion follows as in the corresponding case in Theorem 2.1.2. □

2.5. Minimal and mountain-pass solutions

We devote this section to the proof of Theorem 2.1.1: in order do it, we make full use of the extension technique. We recall that in terms of the s-extension, problem (P_λ^s) can be reformulated as

$$(P_\lambda^*)\qquad \begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(U) = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ U > 0 & \text{on } \Omega \times \{y = 0\}, \\ \dfrac{\partial U}{\partial \nu^s} = f_\lambda(U) & \text{on } \Omega \times \{y = 0\}, \end{cases}$$

where $f_\lambda(s) = \lambda s^q + s^r$. On the other hand, in order to apply the Mountain Pass Theorem by Ambrosetti-Rabinowitz [**12**], we show the PS condition for the energy functional I_λ associated to (P_λ^s), defined in (2.2.1).

LEMMA 2.5.1. *Let $\{u_n\} \subset H^s_{\Sigma_\mathcal{D}}(\Omega)$ be a PS sequence, i.e., $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$. Then, there exist a subsequence (again denoted by) u_n strongly convergent in $H^s_{\Sigma_\mathcal{D}}(\Omega)$.*

PROOF. Since $I_\lambda(u_n) \to c$ it follows that $\|u_n\|_{H^s_{\Sigma_\mathcal{D}}(\Omega)} \leq C$ uniformly for some positive constant. By the compactness of the Sobolev embeddings, there exists a subsequence still denoted by $\{u_n\}$ such that

$$u_n \to u \quad \text{in } L^r(\Omega), \text{ for any } 1 \leq r < 2_s^*,$$

and

$$u_n \rightharpoonup u \quad \text{in } H^s_{\Sigma_\mathcal{D}}(\Omega).$$

Using that $I'_\lambda(u_n) \to 0$ together with the above convergences, we have the strong convergence proving the PS condition. □

PROOF OF THEOREM 2.1.1-(1). Consider the eigenvalue problem associated to the first eigenvalue $\lambda_{1,s}$, and let φ_1 the associated eigenfunction. Using φ_1 as a test function in problem (P_λ^s), we have

$$(\lambda_{1,s} - \lambda)\int_\Omega u\varphi_1 dx = \int_\Omega u^r \varphi_1 dx,$$

and hence necessarily $\lambda < \lambda_{1,s}$. On the other hand, using the fractional Sobolev inequality together with Poincaré inequality we find

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\int_\Omega |(-\Delta)^{s/2} v|^2 dx - \frac{\lambda}{2}\int_\Omega |v|^2 dx - \frac{1}{r+1}\int_\Omega |v|^{r+1} dx \\ &\geq c_1\left(1 - \frac{\lambda}{\lambda_{1,s}}\right)\int_\Omega |(-\Delta)^{s/2} v|^2 dx - c_2\left(\int_\Omega |(-\Delta)^{s/2} v|^2 dx\right)^{(r+1)/2}, \end{aligned}$$

for positive constants c_1, c_2. Therefore, $v = 0$ is a local minimum for the functional I_λ. Moreover, since $I_\lambda(tv) \to -\infty$ for $t \to \infty$ and, because of Lemma 2.5.1, the PS condition is satisfied the functional I_λ satisfies the hypotheses of the Mountain Pass Theorem by Ambrosetti-Rabinowitz [**12**] and we deduce the existence of at least one solution for $0 < \lambda < \lambda_{1,s}$. Finally, the bifurcation result is a consequence of the classical Rabinowitz Theorem [**76**]. □

Next, in order to continue with the proof of Theorem 2.1.1, we establish some preliminary results. Some of these results can be proved for more general nonlinearities $f(u)$, with f at least continuous , satisfying the growth condition $0 \leq f(s) \leq c(1 + |s|^p)$ for some $p > 0$. In such cases we will denote the associated extension problem as (P_f^*).

The first result deals with the sub and supersolutions method, the proof is rather standard and so we omit it.

LEMMA 2.5.2. *Suppose that there exist a subsolution* U_1 *and a supersolution* U_2 *to* (P_f^*), *i.e.* $U_1, U_2 \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ *such that* $B(U_1) \le 0$, $B(U_2) \ge 0$ *on* $\partial_L \mathscr{C}_\Omega$ *and for every* $0 \le \phi \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ *the following inequalities are satisfied:*

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \nabla U_1 \nabla \phi dxdy \le \int_\Omega f(U_1(x,0))\phi(x,0)dx$$
$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \nabla U_2 \nabla \phi dxdy \ge \int_\Omega f(U_2(x,0))\phi(x,0)dx\,,$$

respectively. Assume moreover that $U_1 \le U_2$ *in* $\mathscr{C}_\Omega$. *Then, there also exists a solution* U *verifying* $U_1 \le U \le U_2$ *in* $\mathscr{C}_\Omega$.

Next we show with a comparison result.

LEMMA 2.5.3. *Let* $U_1, U_2 \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ *be respectively a positive subsolution and a positive supersolution of* (P_f^*) *and assume that* $f(t)/t$ *is decreasing for* $t > 0$. *Then* $U_1 \le U_2$ *in* $\mathscr{C}_\Omega$.

PROOF. The proof is similar to the proof of [**10**, Lemma 3.3]. By definition we have, for any positive test functions ϕ_1, ϕ_2 that belong to $H^1_{\Sigma^*_{\mathcal{D}}}(\mathscr{C}_\Omega)$

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \nabla U_1 \nabla \phi_1 dxdy \le \int_\Omega f(u_1)\phi_1(x,0)dx$$
$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \nabla U_2 \nabla \phi_2 dxdy \ge \int_\Omega f(u_2)\phi_2(x,0)dx,$$

where $u_1 = U_1(x,0)$ and $u_2 = U_2(x,0)$. Let $\theta(t)$ be a smooth non-decreasing function such that $\theta(t) = 0$ for $t \le 0$, $\theta(t) = 1$ for $t \ge 1$, set $\theta_\varepsilon(t) = \theta(t/\varepsilon)$, and define the test functions φ_1 and φ_2 as

$$\varphi_1 = U_2 \theta_\varepsilon\left(U_1 - U_2\right), \quad \varphi_2 = U_1 \theta_\varepsilon\left(U_1 - U_2\right).$$

From the above inequalities we obtain

$$\begin{aligned} J_\varepsilon := \kappa_s & \int_{\mathscr{C}_\Omega} y^{1-2s} \left\langle U_1 \nabla U_2 - U_2 \nabla U_1, \nabla(U_1 - U_2)\right\rangle \theta'_\varepsilon\left(U_1 - U_2\right) dxdy \\ & \ge \int_\Omega u_1 u_2 \left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) \theta_\varepsilon\left(u_1 - u_2\right) dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} J_\varepsilon & \le \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \left\langle \nabla U_1, \left(U_1 - U_2\right) \nabla (U_1 - U_2)\right\rangle \theta'_\varepsilon\left(U_1 - U_2\right) dxdy \\ & = \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \left\langle \nabla U_1, \nabla \eta_\varepsilon(U_1 - U_2)\right\rangle dxdy \\ & = \int_\Omega f(u_1)\eta_\varepsilon(u_1 - u_2) dx, \end{aligned}$$

where $\eta'_\varepsilon(t) = t\theta'_\varepsilon(t)$. Since $0 \le \eta_\varepsilon \le \varepsilon$, we find $I_\varepsilon \le c\varepsilon$. Then, letting $\varepsilon \to 0^+$ we conclude

$$\int\limits_{\Omega\cap\{u_1>u_2\}} u_1 u_2 \left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right) dx \le 0.$$

Taking in mind the hypotheses on f, it follows $u_1 \le u_2$ in Ω. The result for the whole cylinder $\mathscr{C}_\Omega$ follows by the maximum principle. $\square$

Next we focus on the remaining assertions in Theorem 2.1.1-(2). Thus, from now on we assume that $0 < q < 1$.

LEMMA 2.5.4. *Let* Λ *defined by*

$$\Lambda = \sup\{\lambda > 0 : (P_\lambda) \text{ has solution}\},$$

then, $0 < \Lambda < \infty$.

PROOF. As for the linear case, consider the eigenvalue problem associated to the first eigenvalue $\lambda_{1,s}$, and let φ_1 the associated eigenfunction. Using φ_1 as a test function in problem (P_λ^s), we have

$$\int_\Omega (\lambda u^q + u^r)\varphi_1 dx = \lambda_{1,s} \int_\Omega u\varphi_1 dx. \tag{2.5.1}$$

Since there exists a constant $c = c(r,q) > 1$ such that $\lambda t^q + t^r > c\lambda^\delta t$ with $\delta = \frac{r}{r-q}$, for any $t > 0$, from (2.5.1) we deduce $c\lambda^\delta < \lambda_{1,s}$ and hence $\Lambda < \infty$. In particular, this also proves that there is no solution of (P_λ^s) for $\lambda > \Lambda$.

In order to prove that $\Lambda > 0$, we prove, by means of the sub and supersolution technique the existence of solution of (P_λ^*) for any small positive λ. Indeed, for $\varepsilon > 0$ small enough, $\underline{U} = \varepsilon E_s[\varphi_1]$ is a subsolution of (P_λ^*). A supersolution can be constructed as an appropiate multiple of the function G solution of

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla G) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(G) = 0 & \text{on } \partial_L \mathscr{C}_\Omega, \\ \dfrac{\partial G}{\partial \nu^s} = 1 & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

Since the trace function $g(x) = G(x,0)$ is a solution of

$$\begin{cases} (-\Delta)^s g = 1 & \text{in } \Omega, \\ B(g) = 0 & \text{on } \partial\Omega, \end{cases}$$

because of Theorem 1.3.4 we have $\|g\|_{L^\infty(\Omega)} < +\infty$. Next, since $0 < q < 1 < r$ we can find $\lambda_0 > 0$ such that for all $0 < \lambda \le \lambda_0$ there exists $M = M(\lambda)$ such that

$$M \ge \lambda M^q \|g\|^q_{L^\infty(\Omega)} + M^r \|g\|^r_{L^\infty(\Omega)}.$$

As a consequence, the function $h = Mg$ satisfies $M = (-\Delta)^s h \ge \lambda h^q + h^r$ and, by the maximum principle, the extension function $\overline{U} = E_s[h]$ is a supersolution and $\underline{U} \le \overline{U}$. Applying Lemma 2.5.2 we conclude the existence of a solution U to problem (P_λ^*). Therefore, the function $u(x) = U(x,0)$ is a solution of problem (P_λ^s), $\lambda < \lambda_0$. $\square$

Remark 2.5.1. *Although Lemma 2.5.4 provides the existence of a solution for small $\lambda > 0$, we can also prove this result studying the associated functional I_λ. Indeed,*

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\int_\Omega |(-\Delta)^{s/2}v|^2 dx - \frac{\lambda}{q+1}\int_\Omega |v|^{q+1}dx - \frac{1}{r+1}\int_\Omega |v|^{r+1}dx \\ &\geq \frac{1}{2}\int_\Omega |(-\Delta)^{s/2}v|^2 dx - \lambda c_1\left(\int_\Omega |(-\Delta)^{s/2}v|^2 dx\right)^{(q+1)/2} \\ &\quad - c_2\left(\int_\Omega |(-\Delta)^{s/2}v|^2 dx\right)^{(r+1)/2}, \end{aligned}$$

for some positive constants c_1 and c_2. Then, for sufficiently small λ, there exist (at least) two solutions to problem (P_λ^s), one given by minimization and another given by the Mountain-Pass Theorem. The proof is rather common, based on the geometry of the function $g(t) = \frac{1}{2}t^2 - \lambda c_1 t^{q+1} - c_2 t^{r+1}$ (see for instance [**12**], [**54**]*).*

Next we show that there exists a solution for every $\lambda \in (0, \Lambda)$.

Lemma 2.5.5. *Problem (P_λ) has at least a positive minimal solution for every $0 < \lambda < \Lambda$. Moreover, the family $\{u_\lambda\}$ of minimal solutions is increasing in λ.*

Proof. By definition of Λ, for any $0 < \lambda < \Lambda$ there exists $\mu \in (\lambda, \Lambda]$ such that (P_μ^*) admits a solution U_μ. It is easy to see that U_μ is a supersolution for (P_λ^*). On the other hand, let V_λ the unique solution of problem (P_f^*) with $f(t) = \lambda t^q$ (the existence can be deduced by minimization, while uniqueness follows from Lemma 2.5.3). It is clear that V_λ is a subsolution of problem (P_λ^*) and, because of Lemma 2.5.3, we have $V_\lambda \leq U_\mu$. Therefore, by Lemma 2.5.2, we conclude that there is a solution of (P_λ^*) and, as a consequence, for the whole open interval $(0, \Lambda)$. Finally, we prove the existence of a minimal solution for all $0 < \lambda < \Lambda$. Indeed, given a solution u to (P_λ^s) we take $U = E_s(u)$ and, by Lemma 2.5.3 being U solution of problem (P_λ^*), it satisfies $V_\lambda \leq U$ with V_λ solution of problem (P_f^*) with $f(t) = \lambda t^q$. Then, the function $v_\lambda(x) = V_\lambda(x, 0)$ is a subsolution of problem (P_λ^s) and the monotone iteration

$$(-\Delta)^s u_{n+1} = \lambda u_n^q + u_n^r, \quad u_n \in H_{\Sigma_{\mathcal{D}}}^s(\Omega) \quad \text{with} \quad u_0 = v_\lambda,$$

verifies $u_n \leq U(x, 0) = u$ and $u_n \nearrow u_\lambda$ with u_λ solution of problem (P_λ^s). In particular $u_\lambda \leq u$ and we conclude that u_λ is a minimal solution. The monotonicity follows directly from first part of the proof, taking $U_\mu = E_s(u_\mu)$ which leads to $u_\lambda \leq u_\mu$ whenever $0 < \lambda < \mu \leq \Lambda$. $\square$

Remark 2.5.2. *Since the number $M = M(\lambda)$ in the proof of Lemma 2.5.4 verifies $M(\lambda) \to 0$ as $\lambda \to 0$, we have $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$.*

Lemma 2.5.6. *Problem (P_λ^*) has at least one solution if $\lambda = \Lambda$.*

To prove Lemma 2.5.6 we extend [**10**, Lemma 3.5] to the fractional framework. This result guarantees that the linearized equation corresponding to (P_λ^s) has non-negative eigenvalues at the minimal solution.

Proposition 2.5.1. *Let u_λ be the minimal solution of (P_λ^s) and define $a_\lambda = a_\lambda(x) = \lambda q u_\lambda^{q-1} + r u_\lambda^{r-1}$. Then, the operator $[(-\Delta)^s - a_\lambda(x)]$ with mixed boundary conditions has a first eigenvalue $\nu_1 \geq 0$.*

REMARK 2.5.3. *In particular, from Proposition 2.5.1, it follows that*

$$\int_\Omega \left(|(-\Delta)^{s/2}v|^2 - a_\lambda v^2\right) dx \geq 0, \quad \text{for all } v \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{2.5.2}$$

PROOF OF PROPOSITION 2.5.1. By contradiction, assume that $\nu_1 < 0$ and let $\phi_1 > 0$ be the first eigenfunction. Let $\alpha > 0$ and observe that since $0 < q < 1$,

$$\begin{aligned}
&(-\Delta)^s(u_\lambda - \alpha\phi_1) - (\lambda(u_\lambda - \alpha\phi_1)^q + (u_\lambda - \alpha\phi_1)^r) \\
&\qquad = \lambda u_\lambda^q + u_\lambda^r - \alpha\nu_1\phi_1 - \alpha\left(\lambda q u_\lambda^{q-1} + r u_\lambda^{r-1}\right)\phi_1 - \lambda(u_\lambda - \alpha\phi_1)^q - (u_\lambda - \alpha\phi_1)^r \\
&\qquad \geq u_\lambda^r - \alpha\nu_1\phi_1 - \alpha r u_\lambda^{r-1}\phi_1 - (u_\lambda - \alpha\phi_1)^r \\
&\qquad = -\alpha\nu_1\phi_1 + O(\alpha\phi_1) \geq 0.
\end{aligned}$$

Hence, $u_\lambda - \alpha\phi_1$ is a supersolution for $\alpha > 0$ small enough.

Now, let ψ a solution of

$$\left\{\begin{array}{ll} (-\Delta)^s v = v^q & \text{in } \Omega, \\ B(v) = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2.5.3}$$

such that $\psi \leq u_\lambda - \alpha\phi_1$ so that problem (P^s_λ) has a solution $\widetilde{u}$ such that $\psi \leq \widetilde{u} \leq u_\lambda - \alpha\phi_1$ in contradiction with the minimality of u_λ. $\square$

PROOF OF LEMMA 2.5.6. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \nearrow \Lambda$ and denote by $u_n = u_{\lambda_n}$ the minimal solution of problem (P_{λ_n}). Let $U_n = E_s[u_n]$, then

$$I_{\lambda_n}(u_n) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}} u_n|^2 dx - \frac{\lambda_n}{q+1}\int_\Omega u_n^{q+1} dx - \frac{1}{r+1}\int_\Omega u_n^{r+1} dx.$$

Moreover, as u_n is a solution of (P^s_λ), it also satisfies

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \lambda_n \int_\Omega u_n^{q+1} dx + \int_\Omega u_n^{r+1} dx.$$

On the other hand, using (2.5.2) with $v = u_n$,

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} u_n|^2 dx - \lambda_n q \int_\Omega u_n^{q+1} dx - r\int_\Omega u_n^{r+1} dx \geq 0.$$

As in [**10**, Lemma 3.5], we conclude $I_{\lambda_n}(u_n) < 0$. Since $I'_{\lambda_n}(u_n) = 0$, it is easy to obtain that $\|u_n\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} \leq C$. Hence, there exists a weakly convergent subsequence $u_n \to u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and, as a consequence, u is a weak solution of (P^s_λ) for $\lambda = \Lambda$. $\square$

The existence of a second solution of (P^s_λ) for every $0 < \lambda < \Lambda$ is proved following the ideas of [**4**], developed to concave-convex problems in [**2, 24**] for the classical Laplacian and the spectral fractional Laplacian respectively.

LEMMA 2.5.7. *Problem (P_λ) has at least two solutions for each $\lambda \in (0, \Lambda)$.*

PROOF. The proof follows exactly as in [**24**, Lemma 5.11] so we omit the details. $\square$

2.5.1. Moving the boundary conditions.

Now we prove Theorem 2.1.4, i.e. the assertions on the behavior of the minimal and mountain pass solutions when we move the boundary conditions (see hypotheses (B_1)-(B_3)). To this aim, we need the following result.

LEMMA 2.5.8. *Let v be the solution of problem* (2.5.3). *There exists a constant $\beta > 0$ such that*

$$\|\phi\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} - q\int_\Omega v^{q-1}\phi^2 dx \geq \beta\|\phi\|^2_{L^2(\Omega)}, \quad \textit{for all } \phi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{2.5.4}$$

PROOF. Since we always consider boundary conditions such that $|\Sigma_{\mathcal{D}}| = \alpha > 0$, the function v can be obtained as

$$\min\left\{\|\phi\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} - \frac{1}{q+1}\|\phi\|^{q+1}_{L^{q+1}(\Omega)} : \phi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\right\},$$

and thus,

$$\|\phi\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)} - q\int_\Omega v^{q-1}\phi^2 dx \geq 0, \quad \text{for all } \phi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega).$$

As a consequence, the linearized problem

$$\left\{\begin{array}{rll} (-\Delta)^s\varphi - qv^{q-1}\varphi = \mu\varphi & & \text{in } \Omega, \\ B(\varphi) = 0 & & \text{on } \partial\Omega, \end{array}\right. \tag{2.5.5}$$

has a non-negative first eigenvalue μ_1. Let φ_1 be the first eigenfunction and assume $\mu_1 = 0$. Since v is a solution of (2.5.3), then

$$q\int_\Omega v^q\varphi_1 dx = \int_\Omega v^q\varphi_1 dx.$$

which is a contradiction. Hence $\mu_1 > 0$. □

LEMMA 2.5.9. *There exists $A > 0$ such that for all $\lambda \in (0, \Lambda)$ the problem (P^s_λ) has at most one solution satisfying $\|u\|_{L^\infty(\Omega)} < A$.*

PROOF. Let $A > 0$ such that $rA^{r-1} < \beta$, with β given by (2.5.4). Assumme by contradiction that there exists a second solution $u = u_\lambda + w$ of (P^s_λ) such that $\|u\|_{L^\infty(\Omega)} \leq A$. Since u_λ is the minimal solution, $w \geq 0$. Let $\zeta(x) = \lambda^{\frac{1}{1-q}}v(x)$ with v the solution of (2.5.3), so that $(-\Delta)^s\zeta = \lambda\zeta^q$. Moreover, u_λ is also a supersolution of (2.5.3), and hence, by Lemma 2.5.3, $u_\lambda \geq \lambda^{\frac{1}{1-q}}v$. On the other hand, since $u = u_\lambda + w$ is a solution of (P^s_λ) we have

$$(-\Delta)^s(u_\lambda + w) = \lambda(u_\lambda + w)^q + (u_\lambda + w)^r.$$

By concavity, $\lambda(u_\lambda + w)^q \leq \lambda u^q_\lambda + \lambda q u^{q-1}_\lambda w$ and hence

$$(-\Delta)^s w \leq \lambda q u^{q-1}_\lambda w + (u_\lambda + w)^r - u^r_\lambda.$$

Furthermore, since $u_\lambda \geq \lambda^{\frac{1}{1-q}}v$, one also has $u^{q-1}_\lambda \leq \lambda^{-1}v^{q-1}$ and as we are assuming $\|u\|_{L^\infty(\Omega)} \leq A$, we find

$$\begin{aligned}(-\Delta)^s w &\leq qv^{q-1} + (u_\lambda + w)^r - u^r_\lambda \\ &\leq qv^{q-1} + rA^{r-1}.\end{aligned}$$

Multiplying the above inequality by w and using (2.5.4) we conclude

$$\beta\int_\Omega w^2dx \leq rA^{r-1}\int_\Omega w^2dx.$$

Since $rA^{r-1}<\beta$, it follows $w=0$. $\square$

Now we can perform the proof of Theorem 2.1.4.

PROOF OF THEOREM 2.1.4. First we claim that if $A=A(\alpha)$ is the associated constant to $(P^s_{\alpha,\lambda})$ obtained in Lemma 2.5.9, then $A(\alpha)\to 0$ as $\alpha\to 0$.

Indeed, it is enough to observe that

$$0<\mu_1\leq \lambda_{1,s}(\alpha)=\inf_{\substack{u\in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\\ u\not\equiv 0}}\frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

where μ_1 is the first eigenvalue of the linearized eigenvalue problem (2.5.5).

Since by definition $\lambda_{1,s}(\alpha)=\lambda_1^s(\alpha)$ and because of [**34**, Lemma 4.1], $\lambda_1^s(\alpha)$ as $\alpha\to 0$, the result follows.

In particular we deduce:

(1) From the proof of Lemma 2.5.4, we have $c\Lambda^\delta(\alpha)<\lambda_{1,s}(\alpha)$ and arguing as above $\Lambda(\alpha)\to 0$ as $\alpha\to 0$.
(2) There exist at most one solution u to (P^s_λ) with $(\lambda,\|u\|_\infty)\in(0,\Lambda(\alpha))\times(0,A(\alpha))$, that is the minimal solution and, since $A(\alpha)\to 0$ as $\alpha\to 0$, converges to zero as $\alpha\to 0$.

Now we prove that for $0<\lambda<\Lambda(\alpha)$ small enough, the solution of problem $(P^s_{\alpha,\lambda})$ obtained by the Mountain Pass Theorem, u_α, satisfies

$$\|u_\alpha\|_{H^s(\Omega)}\to 0,\quad \text{as } \alpha\to 0.$$

The proof follows the lines of [**34**, Lemma 5.12]:

$$\begin{aligned}
I_\lambda(u_\alpha)&=\frac{1}{2}\int_\Omega|(-\Delta)^{\frac{s}{2}}u_\alpha|^2dx-\frac{\lambda}{q+1}\int_\Omega u_\alpha^{q+1}dx-\frac{1}{r+1}\int_\Omega u_\alpha^{r+1}dx\\
&=\frac{1}{2}\|u_\alpha\|^2_{H^s_{\Sigma_{\mathcal{D}}}}-\frac{\lambda}{q+1}\|u_\alpha\|^{q+1}_{L^{q+1}(\Omega)}-\frac{1}{r+1}\|u_\alpha\|^{r+1}_{L^{r+1}(\Omega)}\\
&\geq\frac{1}{2}\|u_\alpha\|^2_{H^s_{\Sigma_{\mathcal{D}}}}-\frac{\lambda}{q+1}|\Omega|^{1-\frac{q+1}{2^*_s}}\left(1+\frac{1}{\lambda_{1,s}(\alpha)}\right)^{\frac{q+1}{2}}\|u_\alpha\|^{q+1}_{H^s_{\Sigma_{\mathcal{D}}}}\\
&\quad-\frac{1}{r+1}|\Omega|^{1-\frac{q+1}{2^*_s}}\left(1+\frac{1}{\lambda_{1,s}(\alpha)}\right)^{\frac{r+1}{2}}\|u_\alpha\|^{r+1}_{H^s_{\Sigma_{\mathcal{D}}}}.
\end{aligned}$$

Let us define $g(t)=\frac{1}{2}t^2-\lambda c_1(q,|\Omega|)\lambda_{1,s}^{-\frac{q-1}{2}}t^{q+1}-c_2(r,|\Omega|)\lambda_{1,s}^{-\frac{r+1}{2}}t^{r+1}$. It is easy to see that if t_α is such that $g'(t_\alpha)=0$ then $t_\alpha\leq c(r,|\Omega|)\lambda^\mu_{1,s}(\alpha)$ with $\mu=\frac{r+1}{2(r-1)}$, so that $t_\alpha\to 0$ as $\alpha\to 0$. Hence, the Mountain Pass solution converges to zero as $\alpha\to 0$.

$\square$

REMARK 2.5.4. *As a conclusion of the above arguments:*

(1) Both solutions, the minimal solution u_λ and the mountain pass solution u_{mp}, converge to zero as $\alpha \to 0$.

(2) If we set $\alpha \in I_\varepsilon = [\varepsilon, |\partial\Omega|]$ with $\varepsilon > 0$ and the hypotheses $(\mathfrak{B}_\alpha)$ and (B_1)-(B_3), there exists $M_\varepsilon, \Lambda_\varepsilon$ such that the family $\mathcal{S}_\varepsilon \subset [0, \Lambda_\varepsilon] \times [0, M_\varepsilon]$.

Part 2

Critical Problems with Mixed Dirichlet-Neumann Boundary data

CHAPTER 3

The Brezis-Nirenberg problem for the fractional Laplacian with mixed Dirichlet-Neumann boundary conditions

We start our study of critical problems with Chapter 3, where we address the existence of solutions to the critical Brezis-Nirenberg problem when one deals with the spectral fractional Laplace operator and mixed Dirichlet-Neumann boundary conditions, i.e.,

$$\left\{\begin{array}{rcl}(-\Delta)^s u &=& \lambda u + u^{2_s^*-1}, \quad u>0 \quad \text{in} \quad \Omega,\\ u &=& 0 \quad \text{on} \quad \Sigma_{\mathcal{D}},\\ \dfrac{\partial u}{\partial \nu} &=& 0 \quad \text{on} \quad \Sigma_{\mathcal{N}},\end{array}\right.$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $\frac{1}{2} < s < 1$, 2_s^* is the critical fractional Sobolev exponent, $0 \leq \lambda \in \mathbb{R}$, ν is the outwards normal to $\partial\Omega$, $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega$, $\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset$, and $\Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma$ is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$.

3.1. Introduction

A turning point in the history of elliptic boundary problems was the seminal paper by Brezis and Nirenberg [**26**], where the critical power problem for the classical Laplacian with a lower-order perturbation term and a Dirichlet boundary condition was studied. For the pure critical problem it is well known that there is no positive solution when the domain is star-shaped due to a Pohozaev identity, cf. [**74**]. Nevertheless, Brezis and Nirenberg proved, among other results, that there exists a positive solution when the perturbation is linear, analyzing more carefully the case when the domain is a ball. Since then, there have arisen more than one thousand papers citing [**26**]. In the fractional setting, Brezis-Nirenberg problems have been also widely investigated. For brevity we just cite some related works dealing only with the fractional Laplacian, cf., e.g. [**18, 84**] for the spectral fractional Laplacian and [**69, 79**] for the fractional Laplacian defined by a singular integral in (1.2.3); both with Dirichlet boundary condition. As we said above, there are no references dealing with problems involving the spectral fractional Laplacian and mixed Dirichlet-Neumann boundary conditions. As a consequence, the main goal of this chapter is twofold: one is to address for the very first time problems involving spectral fractional Laplacian together with mixed Dirichlet-Neumann boundary conditions, and second to prove existence of a positive solution for the Brezis-Nirenberg problem in this fractional setting with mixed boundary conditions.

The precise problem we study in this chapter is the following,

$$(P_\lambda^c) \qquad \left\{\begin{array}{ll} (-\Delta)^s u = \lambda u + u^{2_s^*-1} & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega = \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}}, \end{array}\right.$$

where $\frac{1}{2} < s < 1$, Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 2s$, and mixed Dirichlet-Neumann boundary conditions of the form

$$B(u) = \chi_{\Sigma_{\mathcal{D}}}(x) \cdot u + \chi_{\Sigma_{\mathcal{N}}}(x) \cdot \frac{\partial u}{\partial \nu}, \tag{3.1.1}$$

where ν is the outwards normal to $\partial\Omega$, χ_A stands for the characteristic function of a set A, $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_{\mathcal{D}}$ is a closed submanifold of $\partial\Omega$, with positive Lebesgue measure, say $|\Sigma_{\mathcal{D}}| = \alpha > 0$, $\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset$, $\Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega$ and $\Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma$ is a smooth $(N-2)$-dimensional submanifold.

For the Dirichlet case ($|\Sigma_{\mathcal{N}}| = 0$) it can be seen (see [**24**]) that using a generalized Pohozaev identity, problem (P_λ^c) has no solution for $\lambda = 0$ and Ω a star-shaped domain. As we will see, in the mixed boundary data case the situation is different.

The classical Pohozaev's identity was extended to the mixed Dirichlet-Neumann boundary data case, involving the classical Laplace operator by Lions-Pacella-Tricarico [**67**]. Following those ideas, we extend that result to our mixed fractional setting. Precisely, as in [**2, 34**], we will show that taking the mixed Dirichlet-Neumann boundary conditions, in an appropriate way, problem (P_λ^c) has a solution when $\lambda = 0$, in contrast to the Dirichlet case. Thus, we can include the value $\lambda = 0$ in the existence results. The main result proved in this chapter is the following.

THEOREM 3.1.1. *Assume that $\frac{1}{2} < s < 1$ and $N \geq 4s$. Let $\lambda_{1,s}$ be the first eigenvalue of the fractional operator $(-\Delta)^s$ with mixed Dirichlet-Neumann boundary conditions* (3.1.1)*. Then problem (P_λ^c)*

(1) has no solution for $\lambda \geq \lambda_{1,s}$,
(2) has at least one solution for $0 < \lambda < \lambda_{1,s}$,
(3) has at least one solution for $\lambda = 0$ and $|\Sigma_{\mathcal{D}}|$ small enough.

Note that the range $\frac{1}{2} < s < 1$ is natural for mixed boundary problems in our fractional setting, see Section 1.2.

Organization of the chapter: In Section 3.2 we establish the appropriate functional setting and, using the ideas of [**61**] and [**2**], we also introduce two constants $\widetilde{S}(\Sigma_{\mathcal{N}})$ and $\widetilde{S}(\Sigma_{\mathcal{D}})$ respectively, that play a similar role to that of the Sobolev constant in the celebrated paper of Brezis and Nirenberg, [**26**]. In Section 3.3 we study some useful properties of those constants. Section 3.4 is devoted to prove Theorem 3.1.1 and it is divided into two subsections. In Subsection 3.4.1 we prove the statements (1)-(2) in Theorem 3.1.1. In Subsection 3.4.2, we use the constant $\widetilde{S}(\Sigma_{\mathcal{D}})$ to study the existence of solution to problem (P_λ^c) when we move the boundary conditions in an appropriate way to be specified. These results allow us to prove statement (3) in Theorem 3.1.1. Finally, in the last section we prove a non-existence result by means of a Pohozaev-type identity.

3.2. Functional setting and definitions

Along Chapter 3 we will follow the notation and framework introduced in Chapter 1. We refer to Section 1.2 for the definition of the spectral fractional Laplacian as well as some useful properties about the extension technique exposed there.

DEFINITION 3.2.1. *We say that* $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ *is a solution of* (P^c_λ) *if*

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}\psi dx = \int_\Omega \left(\lambda u + u^{2^*_s-1}\right)\psi dx, \quad \textit{for all } \psi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{3.2.1}$$

The right-hand side of (3.2.1) is well defined because of the embedding $H^s_{\Sigma_{\mathcal{D}}}(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$ while $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ so $\lambda u + u^{2^*_s-1} \in L^{\frac{2N}{N+2s}} \hookrightarrow \left(H^s_{\Sigma_{\mathcal{D}}}(\Omega)\right)'$. The energy functional associated with problem (P^c_λ) is

$$I(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{s/2}u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{N-2s}{2N}\int_\Omega u^{\frac{2N}{N-2s}}dx. \tag{3.2.2}$$

This functional is well defined in $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and critical points of I, defined by (3.2.2), correspond to solutions of (P^c_λ).
In order to overcome some difficulties that appear along several proofs in this chapter, we use the the extension technique as done in previous chapters to give an equivalent definition of the operator $(-\Delta)^s$ defined in a bounded domain by means of an auxiliary problem. Following the arguments in Section 1.2, we can reformulate our problem (P^c_λ) in terms of the extension problem as follows

$$(P^*_\lambda) \qquad \left\{ \begin{array}{ll} -div(y^{1-2s}\nabla w) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(w) = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ \frac{\partial w}{\partial \nu^s} = \lambda w + w^{2^*_s-1} & \text{in } \Omega \times \{y=0\}. \end{array}\right.$$

DEFINITION 3.2.2. *An* energy solution *of problem* (P^*_λ) *is a function* $w \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ *such that*

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\langle \nabla w, \nabla\varphi\rangle dxdy = \int_\Omega \left(\lambda w(x,0) + w^{2^*_s-1}(x,0)\right)\varphi(x,0)dx, \ \forall\, \varphi \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$$

Let us observe that, given $w \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ a solution of problem (P^*_λ), the function $u(x) = Tr[w](x) = w(x,0)$ belongs to the space $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and it is an energy solution of problem (P^c_λ). Also the vice versa is true, if $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ is a solution of (P^c_λ) then $w = E_s[u] \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ is a solution of (P^*_λ) and, as a consequence, both formulations are equivalent. Finally, the energy functional associated with problem (P^*_λ) is the following,

$$J(w) = \frac{\kappa_s}{2}\int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy - \frac{\lambda}{2}\int_\Omega w^2 dx - \frac{N-2s}{2N}\int_\Omega w^{2^*_s}dx. \tag{3.2.3}$$

Plainly, critical points of J in $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ correspond to critical points of I in $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$. Moreover, minima of J also correspond to minima of I. The proof of this fact is similar to the one of the Dirichlet case, see [**18**].

Consider now the following quotient

$$Q_\lambda(w) = \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} - \lambda\|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}},$$

where $w = E_s[u]$, and take

$$S_\lambda(\Omega) = \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)\\ w\not\equiv 0}} \big\{Q_\lambda(w)\big\}. \tag{3.2.4}$$

If the constant $S_\lambda(\Omega)$ is achieved then problem (P^*_λ) will have at least one solution, and thus problem (P^c_λ) has also at least one solution, as we will see in the proof of Theorem 3.1.1. To study the behavior of $Q_\lambda(\cdot)$ we introduce the constants $\widetilde{S}(\Sigma_{\mathcal{N}})$ and $\widetilde{S}(\Sigma_{\mathcal{D}})$ which are inspired by the works [**61**] and [**2**] respectively.

Definition 3.2.3. *For $x_0 \in \Sigma_{\mathcal{N}}$ we define the function*

$$\begin{aligned}\Theta_\lambda \colon \Sigma_{\mathcal{N}} &\to \mathbb{R}\\ x_0 &\mapsto \Theta_\lambda(x_0),\end{aligned}$$

by

$$\Theta_\lambda(x_0) = \lim_{\rho\to 0} S_\lambda\big(\Omega_\rho(x_0)\big),$$

where $\Omega_\rho(x_0) = \Omega \cap B_\rho(x_0)$ and the respective infimum in $S_\lambda\big(\Omega_\rho(x_0)\big)$ is taken over the set of functions that vanish on $\Sigma^\rho_{\mathcal{D}} = \partial\Omega_\rho(x_0)\cap\Omega$.
We define the Sobolev constant relative to the Neumann boundary part as

$$\widetilde{S}(\Sigma_{\mathcal{N}}) = \inf_{x_0\in\Sigma_{\mathcal{N}}} \Theta_\lambda(x_0).$$

This constant plays a major role in the existence issues of problem (P^c_λ), similar of that of the Sobolev constant in the classical Brezis-Nirenberg problem. The next three theorems, which are going to be proved in Section 3.4, will be useful in the proof of the main result, Theorem 3.1.1.

Theorem 3.2.1. *If $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{N}})$ then the infimum* (3.2.4) *is achieved.*

As we will see below, the constant $\widetilde{S}(\Sigma_{\mathcal{N}})$ depends only on the regularity of the Neumann boundary part, but it is independent of the Dirichlet boundary part $\Sigma_{\mathcal{D}}$. Since the properties of a Dirichlet problem are quite different from those of a Neumann problem, one would expect that this fact is reflected when we move our boundary conditions, specifically when $|\Sigma_{\mathcal{D}}| = \alpha \to 0$, see Lemma 3.4.2 below. To do so we define the following constant.

Definition 3.2.4. *The part is defined by*

$$\widetilde{S}(\Sigma_{\mathcal{D}}) = \inf_{\substack{u\in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\\ u\not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}}.$$

REMARK 3.2.1. *As it is noted in the proof of Lemma 1.2.1, the extension function minimizes the* $\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}$ *norm along all the functions with the same trace on* $\{y=0\}$*, thus we can reformulate the definition of* $\widetilde{S}(\Sigma_{\mathcal{D}})$ *as follows,*

$$\widetilde{S}(\Sigma_{\mathcal{D}}) = \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)\\ w\not\equiv 0}} \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)}}.$$

Arguing in a similar way as in [**2**, Theorem 2.2] we can prove the following theorem.

THEOREM 3.2.2. *If* $\widetilde{S}(\Sigma_{\mathcal{D}}) < 2^{\frac{-2s}{N}}\kappa_s S(s,N)$ *then* $\widetilde{S}(\Sigma_{\mathcal{D}})$ *is attained.*

REMARKS 3.2.1. *This result makes the difference between the Dirichlet boundary condition case and the mixed Dirichlet-Neumann boundary condition case.*

(1) *In the Dirichlet case, by taking* $\lambda = 0$ *in* (P^c_λ)*, we have the critical power problem, which has no solution if the domain* Ω *is star-shaped, see* [**24**]. *We mention here that, in the classical case* $s=1$*, many important researches have been devoted to the study of the effect of the domain shape on the existence of solution of the critical problem* (P^c_0) *with Dirichlet boundary data. For example, if* Ω *is star-shaped, Pohozaev, see* [**74**]*, proved that* (P^c_0) *has no solution while Bahri and Coron, see* [**16**]*, proved that if* Ω *has non-trivial topology then* (P^c_0) *has a solution. On the other hand, in* [**31**] *some non-existence results are obtained in bounded domains, which are contractible but not star-shaped, whereas in* [**38**], [**43**] *and* [**73**] *the existence of contractible bounded domains* Ω *where* (P^*_0) *has solution is proved.*
(2) *In the mixed case, the corresponding Sobolev constant* $\widetilde{S}(\Sigma_{\mathcal{D}})$ *can be achieved thanks to Theorem 3.2.2. As we will see, the hypotheses of Theorem 3.2.2 can be fulfilled by moving the size of the Dirichlet boundary part.*

The next result is analogous to that of Theorem 3.2.1 for the constant relative to the Dirichlet part.

THEOREM 3.2.3. *If* $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ *then* $S_\lambda(\Omega)$ *is attained.*

3.3. Properties of the constants $\widetilde{S}(\Sigma_{\mathcal{N}})$ and $\widetilde{S}(\Sigma_{\mathcal{D}})$

PROPOSITION 3.3.1. *The constant* $\widetilde{S}(\Sigma_{\mathcal{N}})$ *does not depend on* λ*, moreover, if* $\Sigma_{\mathcal{N}}$ *is a regular* $(N-1)$*-dimensional submanifold of* $\partial\Omega$*, then* $\widetilde{S}(\Sigma_{\mathcal{N}}) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$.

We split the proof into several Lemmas.

LEMMA 3.3.1. *The constant* $\widetilde{S}(\Sigma_{\mathcal{N}})$ *does not depend on* λ.

PROOF. Note that by the very definition of $\widetilde{S}(\Sigma_{\mathcal{N}})$ it is enough to prove that $\Theta_\lambda(x_0)$ does not depend on λ, that is $\Theta_\lambda(x_0) = \Theta(x_0) = \lim\limits_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big)$. Since $\lambda \geq 0$, then it is immediate that $\Theta_\lambda(x_0) \leq \lim\limits_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big)$. On the other hand, using Hölder's inequality and the trace inequality (1.2.9) jointly, we get

$$\|\varphi\|^2_{L^2(\Omega_\rho)} \leq |\Omega_\rho(x_0)|^{\frac{2s}{N}}\|\varphi\|^2_{L^{2^*_s}(\Omega_\rho(x_0))} \leq C|\Omega_\rho(x_0)|^{\frac{2s}{N}}\|E_s[\varphi]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{\Omega_\rho(x_0)})},$$

thus

$$\Theta_\lambda(x_0) \geq \lim_{\rho\to 0} \left(1 - \lambda C|\Omega_\rho(x_0)|^{\frac{2s}{N}}\right) S_0\big(\Omega_\rho(x_0)\big).$$

And the result follows. $\square$

Bearing in mind Lemma 3.3.1, to prove the last assertion of Proposition 3.3.1, we need to estimate $S_0\big(\Omega_\rho(x_0)\big) = \inf\left\{Q_0(w) : w \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_{\Omega_\rho(x_0)})\right\}$. To do so, we use the family of extremal functions of the Sobolev inequality,

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}},$$

and its s-extension, $w_\varepsilon(x) = E_s[u_\varepsilon]$, times a cut-off function as a test function. Note that both functions u_ε and the Poisson kernel (1.2.3) are self-similar functions, $u_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} u_1(x)$, and $P^s_y(x) = \frac{1}{y^N} P^s_1\left(\frac{x}{y}\right)$ so the extension family $w_\varepsilon = E_s[u_\varepsilon]$ satisfies

$$w_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right). \tag{3.3.1}$$

Consider a smooth non-increasing cut-off function $\phi_0(t) \in \mathcal{C}^\infty(\mathbb{R}_+)$, satisfying $\phi_0(t) = 1$ for $0 \leq t \leq \frac{1}{2}$ and $\phi_0(t) = 0$ for $t \geq 1$, and $|\phi_0'(t)| \leq C$ for any $t \geq 0$. Assume, without loss of generality, that $0 \in \Omega$, and define, for some $\rho > 0$ small enough such that $B^+_\rho \subseteq \mathscr{C}_\Omega$, the function $\phi_\rho(x,y) = \phi_0(\frac{r_{xy}}{\rho})$ with $r_{xy} = |(x,y)| = (|x|^2 + y^2)^{\frac{1}{2}}$.

Lemma 3.3.2. *The family $\{\phi_\rho w_\varepsilon\}$ and its trace on $\{y = 0\}$, $\{\phi_\rho u_\varepsilon\}$, satisfy*

$$\|\phi_\rho w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)} = \|w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)} + O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right), \tag{3.3.2}$$

and

$$\int_\Omega |\phi_\rho u_\varepsilon|^{2^*_s} dx = \|u_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} + O\left(\left(\frac{\varepsilon}{\rho}\right)^N\right). \tag{3.3.3}$$

The proof of this Lemma is similar to the proof of [**18**, Lemma 3.8] for the Dirichlet boundary conditions. Note that in the Dirichlet case it is not necessary to control the role of the radius of the cut-off function, on the contrary, in the mixed case, by the very definition of the constant $\widetilde{S}(\Sigma_\mathcal{N})$, a careful analysis of the role of that radius is needed. Now we estate the following result proved in [**18**, Lemma 3.7] that will be useful in the proof of Lemma 3.3.2.

Lemma 3.3.3. [**18**, Lemma 3.7] *The family $w_\varepsilon = w_{\varepsilon,s} = E_s[u_\varepsilon]$ satisfies*

$$|\nabla w_{1,s}(x,y)| \leq C w_{1,s-\frac{1}{2}}(x,y),\ \frac{1}{2} < s < 1,\ (x,y) \in \mathbb{R}^{N+1}_+. \tag{3.3.4}$$

Proof of Lemma 3.3.2. We start with the proof of (3.3.3),

$$\begin{aligned}\int_\Omega |\phi_\rho u_\varepsilon|^{2^*_s} dx &= \int_{\mathbb{R}^N} |\phi_\rho u_\varepsilon|^{2^*_s} dx \geq \int_{|x|<\frac{\rho}{2}} |u_\varepsilon|^{2^*_s} dx \\ &= \|u_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} - \int_{|x|>\frac{\rho}{2}} |u_\varepsilon|^{2^*_s} dx.\end{aligned}$$

Observe that

$$\begin{aligned}\int_{|x|>\frac{\rho}{2}} |u_\varepsilon|^{2_s^*} dx &= \varepsilon^{-N} \int_{|x|>\frac{\rho}{2}} \frac{1}{\left(1+\left(\frac{|x|}{\varepsilon}\right)^2\right)^N} dx = \varepsilon^{-N} \int_{\frac{\rho}{2}}^{\infty} \frac{t^{N-1}}{\left(1+\left(\frac{t}{\varepsilon}\right)^2\right)^N} dt \\ &= \int_{\frac{\rho}{2\varepsilon}}^{\infty} \frac{s^{N-1}}{(1+s^2)^N} ds \le \int_{\frac{\rho}{2\varepsilon}}^{\infty} s^{-N-1} ds = \left(\frac{\varepsilon}{\rho}\right)^N,\end{aligned}$$

so we get

$$\int_\Omega |\phi_\rho u_\varepsilon|^{2_s^*} dx \ge \|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} + O\left(\left(\frac{\varepsilon}{\rho}\right)^N\right).$$

We continue with the proof of (3.3.2). The product $\phi_\rho w_\varepsilon$ satisfies

$$\begin{aligned}\|\phi_\rho w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} &\le \|w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \\ &+ \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} |w_\varepsilon \nabla \phi_\rho|^2 dxdy + 2\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle w_\varepsilon \nabla\phi_\rho, \phi_\rho \nabla w_\varepsilon\rangle dxdy.\end{aligned} \tag{3.3.5}$$

The first term of the right-hand side in (3.3.5) can be estimated as follows,

$$\begin{aligned}\int_{\mathscr{C}_\Omega} y^{1-2s} |w_\varepsilon \nabla\phi_\rho|^2 dxdy &\le \frac{C}{\rho^2} \int_{\{\frac{\rho}{2} \le r_{xy} \le \rho\}} y^{1-2s} w_\varepsilon^2 dxdy \\ &\le \frac{C}{\rho^2} \varepsilon^{N-2s} \int_{\{\frac{\rho}{2} \le r_{xy} \le \rho\}} y^{1-2s} r_{xy}^{-2(N-2s)} dxdy \\ &\le \frac{C}{\rho^2} \varepsilon^{N-2s} \int_{\frac{\rho}{2}}^{\rho} s^{1+2s-N} ds \\ &= O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right),\end{aligned}$$

since $0 \le u_\varepsilon(x) \le \varepsilon^{\frac{N-2s}{2}} |x|^{-(N-2s)}$ and the extension of the function $K(x) = |x|^{-(N-2s)}$ is $\tilde{K}(x,y) = (|x|^2 + y^2)^{-\frac{N-2s}{2}} = r_{xy}^{-(N-2s)}$.
We end with the estimate of the second term of the right-hand side in (3.3.5). Applying

Cauchy-Schwarz inequality and using (3.3.1) we get,

$$
\begin{aligned}
&\int_{\mathscr{C}_\Omega} y^{1-2s}\langle w_\varepsilon \nabla\phi_\rho, \phi_\rho \nabla w_\varepsilon\rangle dxdy \\
&\le \frac{C}{\rho}\int\limits_{\{\frac{\rho}{2}\le r_{xy}\le\rho\}} y^{1-2s}|w_\varepsilon(x,y)||\nabla w_\varepsilon(x,y)|dxdy \\
&\le \frac{C}{\rho}\varepsilon^{-(N-2s)-1}\int\limits_{\{\frac{\rho}{2}\le r_{xy}\le\rho\}} y^{1-2s}\left|w_1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|\left|\nabla w_1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|dxdy \\
&= \frac{C}{\rho}\varepsilon\int\limits_{\{\frac{\rho}{2\varepsilon}\le r_{xy}\le\frac{\rho}{\varepsilon}\}} y^{1-2s}|w_1(x,y)||\nabla w_1(x,y)|dxdy.
\end{aligned}
\tag{3.3.6}
$$

Note that for $(x,y)\in\{\frac{\rho}{2\varepsilon}\le r_{xy}\le\frac{\rho}{\varepsilon}\}$ we have

$$
\begin{aligned}
w_1(x,y) &= \int\limits_{|z|<\frac{\rho}{4\varepsilon}} P_y^s(x-z)u_1(z)dz + \int\limits_{|z|>\frac{\rho}{4\varepsilon}} P_y^s(x-z)u_1(z)dz \\
&\le C\left(\frac{\varepsilon}{\rho}\right)^{N+2s} y^{2s}\int\limits_{|z|<\frac{\rho}{4\varepsilon}} u_1(z)dz + C\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\int\limits_{|z|>\frac{\rho}{4\varepsilon}} P_y^s(x-z)dz \\
&\le C\left(\frac{\varepsilon}{\rho}\right)^{N+2s} y^{2s}\int\limits_{|z|<\frac{\rho}{4\varepsilon}} \frac{1}{|z|^{N-2s}}dz + C\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\int_{\mathbb{R}^N} P_y^s(x-z)dz \\
&\le C\left(\frac{\varepsilon}{\rho}\right)^{N} y^{2s} + C\left(\frac{\varepsilon}{\rho}\right)^{N-2s} \le C\left(\frac{\varepsilon}{\rho}\right)^{N-2s}.
\end{aligned}
\tag{3.3.7}
$$

Using (3.3.7), (3.3.6) and (3.3.4), we get

$$
\begin{aligned}
&\int_{\mathscr{C}_\Omega} y^{1-2s}\langle w_\varepsilon \nabla\phi_\rho, \phi_\rho \nabla w_\varepsilon\rangle dxdy \\
&\le \frac{C}{\rho}\varepsilon\int\limits_{\{\frac{\rho}{2\varepsilon}\le r_{xy}\le\frac{\rho}{\varepsilon}\}} y^{1-2s}\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\left(\frac{\varepsilon}{\rho}\right)^{N-2(s-1/2)} dxdy \\
&\le c\left(\frac{\varepsilon}{\rho}\right)^{2(1+N-2s)}\int\limits_{\{\frac{\rho}{2\varepsilon}\le r_{xy}\le\frac{\rho}{\varepsilon}\}} y^{1-2s}dxdy = O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right).
\end{aligned}
$$

And the proof is complete. □

LEMMA 3.3.4. *Suppose that $\Sigma_{\mathcal{N}}$ is a regular submanifold of $\partial\Omega$, then given $x_0\in\Sigma_{\mathcal{N}}$ it is satisfied that $\Theta_\lambda(x_0) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$.*

PROOF. From Lemma 3.3.1 we know that $\Theta_\lambda(x_0) = \Theta(x_0) = \lim_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big)$, also since $\Sigma_{\mathcal{N}}$ is a regular submanifold of $\partial\Omega$, given $x_0 \in \Sigma_{\mathcal{N}}$ we have that,

$$\lim_{\rho\to 0} \frac{|B_\rho(x_0)\cap\Omega|}{|B_\rho(x_0)|} = \frac{1}{2}. \tag{3.3.8}$$

On the other hand, since w_ε is a minimizer of $S(s,N)$, we have

$$S(s,N) = \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_\varepsilon|^2 dxdy}{\|u_\varepsilon\|^2_{L^{2^*_s}(\mathbb{R}^N)}}.$$

We take now a cut-off function centered at $x_0 \in \Sigma_{\mathcal{N}}$, namely, we take $\psi_\rho(x,y) = \phi_0(\frac{\overline{r}_{xy}}{\rho})$ with $\overline{r}_{xy} = |(x-x_0,y)| = (|x-x_0|^2+y^2)^{\frac{1}{2}}$. Note that $\psi_\rho u_\varepsilon \equiv 0$ on $\partial\Omega_\rho\cap\Omega$. Thanks to (3.3.2) and (3.3.3) we can choose $\varepsilon = \rho^\alpha$ with $\alpha > 1$ such that

$$\|\phi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} = \|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + O\left(\rho^{(\alpha-1)(N-2s)}\right), \tag{3.3.9}$$

and

$$\|\phi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega)} = \|u_\rho\|^2_{L^{2^*_s}(\mathbb{R}^N)} + O\left(\rho^{(\alpha-1)N}\right), \tag{3.3.10}$$

where ϕ_ρ is the same cut-off function of Lemma 3.3.2. Using (3.3.8)-(3.3.10), we have that

$$\begin{aligned}
\Theta(x_0) = \lim_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big) &\le \lim_{\rho\to 0} \frac{\|\psi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{\Omega_\rho(x_0)})}}{\|\psi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega_\rho(x_0))}} \\
&= \lim_{\rho\to 0} \frac{\frac{1}{2}\|\phi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}}{\frac{1}{2^{\frac{2}{2^*_s}}}\|\phi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega)}} \\
&= 2^{\frac{-2s}{N}} \lim_{\rho\to 0} \frac{\kappa_s S(s,N) + O(\rho^{(\alpha-1)(N-2s)})}{1+O(\rho^{(\alpha-1)N})} \\
&= 2^{\frac{-2s}{N}} \kappa_s S(s,N).
\end{aligned}$$

Finally, we focus on the proof of inequality $\Theta(x_0) \ge 2^{\frac{-2s}{N}}\kappa_s S(s,N)$. To this end we assert the following.

Claim: For $x_0 \in \Sigma_{\mathcal{N}}$ we have

$$\Theta_\lambda(x_0) = \Theta(x_0) = \lim_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big) \ge S_0(B_1^+), \tag{3.3.11}$$

where B_1^+ is the half ball of radius 1 centered at x_0 with the Neumann boundary part on the flat part of B_1^+ and the Dirichlet boundary part on the closure of the remaining boundary. To prove the claim, we can argue in a similar way as in [**61**]. If (3.3.11) is not true, there exists $\epsilon > 0$, $r_0 > 0$, such that for $0 < \rho < r_0$ there exists a function $w_\rho \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{\Omega_\rho})$ with $u_\rho = Tr[w_\rho]$ such that

$$\frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{\Omega_\rho})}}{\|u_\rho\|^2_{L^{2^*_s}(\Omega_\rho)}} < S_0(B_1^+) - \epsilon. \tag{3.3.12}$$

Since x_0 is a regular point, there exists a diffeomorfism T_ρ between Ω_ρ and B_ρ^+ such that, for ρ small enough, $T_\rho(\Sigma_{\mathcal{D}}^\rho) = \partial B_\rho^+ \cap \partial B(x_0,\rho)$ and T_ρ transforms the Neumann part of the boundary, $\partial\Omega_\rho \cap \Sigma_{\mathcal{N}}$, into the flat part of B_1^+. Then the function $v_\rho = T_\rho(w_\rho)$ belongs to $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B_\rho^+})$ and

$$\frac{\|v_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_\rho^+})}}{\|v_\rho(x,0)\|^2_{L^{2^*_s}(B_\rho^+)}} \le C_\rho \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{\Omega_\rho})}}{\|u_\rho\|^2_{L^{2^*_s}(\Omega_\rho)}},$$

where C_ρ depends on the diffeomorfism T_ρ and, by the definition of regular point, it can be chosen in such a way that $C_\rho \to 1$ as $\rho \to 0$. Then, for ρ small enough, by (3.3.12) we have

$$\inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_\rho^+})\\ w\not\equiv 0}} \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_\rho^+})}}{\|u_\rho\|^2_{L^{2^*_s}(B_\rho^+)}} < S_0(B_1^+),$$

which is a contradiction because, due to the invariance under scaling, we have

$$\inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_\rho^+})\\ w\not\equiv 0}} \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_\rho^+})}}{\|u_\rho\|^2_{L^{2^*_s}(B_\rho^+)}} = \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_1^+})\\ w\not\equiv 0}} \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_{B_1^+})}}{\|u_\rho\|^2_{L^{2^*_s}(B_1^+)}} = S_0(B_1^+).$$

Finally, by (3.3.2)-(3.3.3) in Lemma 3.3.2 it follows that $S_0(B_1^+) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$ and hence $\Theta(x_0) \ge 2^{\frac{-2s}{N}}\kappa_s S(s,N)$. □

Proof of Proposition 3.3.1. As a consequence of the previous Lemmata we get that if $\Sigma_{\mathcal{N}}$ is a regular submanifold of $\partial\Omega$ then $\widetilde{S}(\Sigma_{\mathcal{N}}) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$. □

We now turn our attention to the Sobolev constant relative to the Dirichlet part of the boundary $\widetilde{S}(\Sigma_{\mathcal{D}})$. We give an estimate for $\widetilde{S}(\Sigma_{\mathcal{D}})$ similar to that of $\widetilde{S}(\Sigma_{\mathcal{N}})$ in Proposition 3.3.1.

Proposition 3.3.2. $\widetilde{S}(\Sigma_{\mathcal{D}}) \le 2^{\frac{-2s}{N}}\kappa_s S(s,N)$.

Proof. To obtain this estimate we use the extremal functions of the Sobolev inequality and proceed in a similar way as in Proposition 3.3.1. The lower bound in Proposition 3.3.1 is due to the fact that the infimum $\widetilde{S}(\Sigma_{\mathcal{N}})$ is taken in the set $\Omega_\rho(x_0)$, on the contrary, for the constant $\widetilde{S}(\Sigma_{\mathcal{D}})$, we do not have such a lower bound by the very definition of $\widetilde{S}(\Sigma_{\mathcal{D}})$. □

3.4. Proof of main results

3.4.1. Proof of Theorem 3.1.1.(1)-(2).

In this subsection we carry out the proof of Theorems 3.2.1, 3.2.2 and 3.2.3 which will be useful in the proof of Theorem 3.1.1.(1)-(2).

We begin with the upper bound of the parameter λ, i.e., statement (1) in Theorem 3.1.1.

LEMMA 3.4.1. *Problem (P_λ) has no solution for $\lambda \geq \lambda_{1,s}$, with $\lambda_{1,s}$ the first eigenvalue of $(-\Delta)^s$ with mixed boundary condition.*

PROOF. Assume that u is solution of (P_λ) and let φ_1 be a positive first eigenfunction of $(-\Delta)^s$. Taking φ_1 as a test function for (P_λ^c) we obtain

$$\lambda_{1,s}\int_\Omega u\varphi_1 dx = \int_\Omega (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi_1 dx = \int_\Omega \left(\lambda u + u^{2_s^*-1}\right)\varphi_1 dx > \lambda\int_\Omega u\varphi_1 dx.$$

Therefore, $\lambda < \lambda_{1,s}$. □

PROPOSITION 3.4.1. *Assume that $0<\lambda<\lambda_{1,s}$. Then $S_\lambda(\Omega) < 2^{\frac{-2s}{N}}\kappa_s S(s,N) = \widetilde{S}(\Sigma_{\mathcal{N}})$.*

PROOF. We recall the following asymptotic identities given in [**18**, Lemma 3.8],

$$\|\phi_r u_\varepsilon\|_{L^2(\Omega)}^2 = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if } N>4s, \\ C\varepsilon^{2s}\log(1/\varepsilon) + O(\varepsilon^{2s}) & \text{if } N=4s, \end{cases} \tag{3.4.1}$$

for some constant $C>0$, ε small enough and ϕ_r a cut-off function similar to the one in Lemma 3.3.2. Proceeding in a similar way as in Proposition 3.3.1, we take a cut-off function centered at a point $x_0 \in \overline{\Sigma}_{\mathcal{N}}$, then using (3.3.2)-(3.3.3) and (3.4.1) jointly, we have the following:

- If $N>4s$,

$$\begin{aligned} Q_\lambda(\phi_r w_\varepsilon) &\leq 2^{\frac{-2s}{N}}\frac{\kappa_s S(s,N) - \lambda C\varepsilon^{2s}\|u_\varepsilon\|_{L^{2_s^*}(\Omega)}^{-2} + O(\varepsilon^{N-2s})}{1+O(\varepsilon^N)} \\ &\leq 2^{\frac{-2s}{N}}\kappa_s S(s,N) - \lambda C\varepsilon^{2s}\|u_\varepsilon\|_{L^{2_s^*}(\Omega)}^{-2} + O(\varepsilon^{N-2s}) \\ &< 2^{\frac{-2s}{N}}\kappa_s S(s,N). \end{aligned}$$

- If $N=4s$ a similar procedure proves that for ε small enough,

$$Q_\lambda(\phi_r w_\varepsilon) \leq 2^{\frac{-2s}{N}}\kappa_s S(s,N) - \lambda C\varepsilon^{2s}\log(1/\varepsilon)\|u_\varepsilon\|_{L^{2_s^*}(\Omega)}^{-2} + O(\varepsilon^{2s}) < 2^{\frac{-2s}{N}}\kappa_s S(s,N).$$

□

Now we enunciate a concentration-compactness result adapted to our fractional setting with mixed boundary conditions. The proof is a minor variation of that of the concentration-compactness result in [**18**, Theorem 5.1], which is an adaptation to the fractional setting with Dirichlet boundary conditions of the classical concentration-compactness technique of P.L. Lions, [**65**]. For the mixed boundary data case involving the classical Laplace operator and Caffarelli-Kohn-Nirenberg weights, [**28**], a concentration-compactness theorem was proved in [**2**]. First, we recall the concept of a tight sequence.

DEFINITION 3.4.1. *We say that a sequence $\{y^{1-2s}|\nabla w_n|^2\}_{n\in\mathbb{N}} \subset L^1(\mathscr{C}_\Omega)$ is tight if for any $\eta>0$ there exists $\rho>0$ such that*

$$\int_{\{y>\rho\}}\int_\Omega y^{1-2s}|\nabla w_n|^2 dxdy \leq \eta,\ \forall n\in\mathbb{N}. \tag{3.4.2}$$

THEOREM 3.4.1 (Concentration-Compactness). *Let $\{w_n\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ be a weakly convergent sequence to w in $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ such that $\{y^{1-2s}|\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight. Let us denote $u_n = Tr[w_n]$, $u = Tr[w]$ and let μ, ν be two nonnegative measures such that*

$$y^{1-2s}|\nabla w_n|^2 \to \mu, \text{ and } |u_n|^{2^*_s} \to \nu, \tag{3.4.3}$$

in the sense of measures. Then, there exist an at most countable set I and points $\{x_i\}_{i\in I} \subset \overline{\Omega}$ such that

(1) $\nu = |u|^{2^*_s} + \sum\limits_{i\in I} \nu_i \delta_{x_i},\ \nu_i > 0,$

(2) $\mu = y^{1-2s}|\nabla w|^2 + \sum\limits_{i\in I} \mu_i \delta_{x_i},\ \mu_i > 0,$

(3) $\mu_i \geq \widetilde{S}(\Sigma_{\mathcal{D}}) \nu_i^{\frac{2}{2^*_s}}.$

Using Theorem 3.4.1 we prove the next result that is analogous to [**67**, Theorem 2.2].

THEOREM 3.4.2. *Let w_m be a minimizing sequence of $S_\lambda(\Omega)$. Then either w_m is relatively compact or the weak limit, $w \equiv 0$. Even more, in the latter case there exist a subsequence w_m and a point $x_0 \in \overline{\Sigma}_{\mathcal{N}}$ such that*

$$y^{1-2s}|\nabla w_m|^2 \to S_\lambda(\Omega)\delta_{x_0}, \text{ and } |u_m|^{2^*_s} \to \delta_{x_0}, \tag{3.4.4}$$

with $u_m = Tr[w_m]$.

PROOF. Since $0 \leq \lambda < \lambda_{1,s}$ it follows that $0 < S_\lambda(\Omega) \leq \widetilde{S}(\Sigma_{\mathcal{D}})$. We distinguish two cases, depending upon if $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ or $S_\lambda(\Omega) = \widetilde{S}(\Sigma_{\mathcal{D}})$:

(1) $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$. In this case we can argue in a similar way as in [**18**, Prop. 4.2] which in turn is based on the technique of Brezis-Nirenberg.

Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ be a minimizing sequence of $S_\lambda(\Omega)$, and suppose without loss of generality that $w_m \geq 0$ and $\|w_m(\cdot,0)\|_{L^{2^*_s}(\Omega)} = 1$. Clearly, this implies that

$$\|w_m\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \leq M, \tag{3.4.5}$$

then, there exists a subsequence (denoted also by $\{w_m\}$) verifying,

$$\begin{aligned} w_m &\rightharpoonup w \text{ weakly in } \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega),\\ w_m(\cdot,0) &\to w(\cdot,0) \text{ strongly in } L^q(\Omega),\ 1 \leq q < 2^*_s,\\ w_m(\cdot,0) &\to w(\cdot,0) \text{ a.e. in } \Omega. \end{aligned}$$

Using the weak convergence we get

$$\begin{aligned} \|w_m\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} &= \|w_m - w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} \\ &\quad + \|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + 2\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\langle \nabla w, \nabla w_m - \nabla w\rangle dxdy \\ &= \|w_m - w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + \|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} + o(1). \end{aligned}$$

Hence,

$$
\begin{aligned}
Q_\lambda(w_m) &= \|w_m\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)} \\
&= \|w_m - w\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)} + \|w\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)} + o(1) \\
&\geq \widetilde{S}(\Sigma_\mathcal{D})\|w_m(\cdot,0) - w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda(\Omega)\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + o(1).
\end{aligned}
$$

Thus, because of the normalization $\|w_m(\cdot,0)\|_{L^{2^*_s}} = 1$, it follows

$$
Q_\lambda(w_m) \geq (\widetilde{S}(\Sigma_\mathcal{D}) - S_\lambda(\Omega))\|w_m(\cdot,0) - w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda(\Omega) + o(1).
$$

Since $\{w_m\}$ is a minimizing sequence of $S_\lambda(\Omega)$, we obtain

$$
o(1) + S_\lambda(\Omega) \geq (\widetilde{S}(\Sigma_\mathcal{D}) - S_\lambda(\Omega))\|w_m(\cdot,0) - w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda(\Omega) + o(1).
$$

Finally, using that $S_\lambda(\Omega) < \widetilde{S}(\Sigma_\mathcal{D})$ it follows

$$
w_m(\cdot,0) \to w(\cdot,0) \text{ in } L^{2^*_s}(\Omega).
$$

By a standard lower semi-continuity argument, w is a minimizer for $Q_\lambda(\cdot)$, so we get that the sequence is relatively compact.

(2) $S_\lambda(\Omega) = \widetilde{S}(\Sigma_\mathcal{D})$. Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathscr{C}_\Omega)$ be a minimizing sequence of $S_\lambda(\Omega)$, and suppose without loss of generality that $w_m \geq 0$ and $\|w_m(\cdot,0)\|_{L^{2^*_s}(\Omega)} = 1$. Thus $\{w_m\}$ is also a minimizing sequence for $\widetilde{S}(\Sigma_\mathcal{D})$ and we proceed in a similar way as in [**67**, Theorem 2.2]. Using Theorem 3.4.1, we get that either $\{w_m\}$ is relatively compact or the weak limit $w \equiv 0$. In the first case, $w \not\equiv 0$, by Theorem 3.4.1 we have

$$
\widetilde{S}(\Sigma_\mathcal{D}) = \int_{\mathscr{C}_\Omega} d\mu \geq \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy + \widetilde{S}(\Sigma_\mathcal{D}) \sum_{i\in I} \nu_i^{\frac{2}{2^*_s}},
$$

as well as

$$
1 = \int_\Omega d\nu = \int_\Omega |u|^{2^*_s} dx + \sum_{i\in I} \nu_i.
$$

By the two expressions above,

$$
\begin{aligned}
\left(\int_\Omega |u|^{2^*_s} dx\right)^{\frac{2}{2^*_s}} &= \left(1 - \sum_{i\in I}\nu_i\right)^{\frac{2}{2^*_s}} && (3.4.6)\\
&\leq \frac{1}{\widetilde{S}(\Sigma_\mathcal{D})} \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy && (3.4.7)\\
&\leq \frac{1}{\widetilde{S}(\Sigma_\mathcal{D})} \left(\widetilde{S}(\Sigma_\mathcal{D}) - \widetilde{S}(\Sigma_\mathcal{D}) \sum_{i\in I} \nu_i^{\frac{2}{2^*_s}}\right) \\
&= 1 - \sum_{i\in I} \nu_i^{\frac{2}{2^*_s}},
\end{aligned}
$$

hence, $\nu_i \leq 1\ \forall i \in I$. And therefore, by (3.4.6) the only possibility is $\nu_i = 0$ for all $i \in I$. This leads to

$$\int_\Omega |u_m|^{2_s^*} dx \to \int_\Omega |u|^{2_s^*} dx,$$

from which we deduce that u_m (and thus $w_m = E_s[u_m]$) is relatively compact.

Now we consider the case $w \equiv 0$ (and thus $u \equiv 0$). In this case by Theorem 3.4.1 and (3.4.6) we get

$$\sum_{i\in I} \nu_i = 1, \text{ and } \sum_{i\in I} \nu_i^{\frac{2}{2_s^*}} \leq 1,$$

then we infer that I must be a singleton, i.e.,

$$\nu = \delta_{x_0} \qquad \text{and} \qquad \mu = \widetilde{S}(\Sigma_{\mathcal{D}})\delta_{x_0} = S_\lambda(\Omega)\delta_{x_0},$$

with $x_0 \in \overline{\Omega}$.

To show that $x_0 \in \overline{\Sigma}_{\mathcal{N}}$ we argue by contradiction. If $x_0 \in \Omega \cup \Sigma_{\mathcal{D}}$, we set $\overline{\phi}_r(x, y)$ as a cut-off function centered at $x_0 \in \Omega$, and define the sequence

$$w_{m,r} = w_m \overline{\phi}_r(x, y),$$

and the traces sequence $\{u_{m,r}\} = \{Tr[w_{m,r}]\}$. Then for all $r > 0$

$$\lim_{m\to\infty} \frac{\kappa_s \displaystyle\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla w_{m,r}|^2 dxdy}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} = \widetilde{S}(\Sigma_{\mathcal{D}}). \tag{3.4.8}$$

Note that for r sufficiently small, the sequence $\{w_{m,r}\}$ belongs to $\mathcal{X}_0^s(\mathscr{C}_\Omega)$, then for any $m \in \mathbb{N}$, by Proposition 3.3.2,

$$\begin{aligned}
\lim_{r\to 0} \frac{\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla w_{m,r}|^2 dxdy}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} &\geq \inf_{\substack{v\in \mathcal{X}_0^s(\mathscr{C}_\Omega)\\ v\not\equiv 0}} \frac{\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla v|^2 dxdy}{\|v(x,0)\|^2_{L^{2_s^*}(\Omega)}} \\
&= \kappa_s S(s, N) \\
&> 2^{\frac{-2s}{N}} \kappa_s S(s, N) \\
&\geq \widetilde{S}(\Sigma_{\mathcal{D}}),
\end{aligned}$$

and we reach a contradiction with (3.4.8). Therefore, $x_0 \in \partial\Omega$. If $x_0 \in \mathring{\Sigma}_{\mathcal{D}}$ arguing as before we reach the same contradiction. As a consequence, $x_0 \in \overline{\Sigma}_{\mathcal{N}}$.

It only remains to prove the tightness condition (3.4.2) for the minimizing sequence $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$, i.e., there is no evanescence. Since $\{w_m\}$ is a minimizing sequence of $S_\lambda(\Omega)$ then $\{w_m\}$ or a multiple will converge to a critical point of the functional (3.2.3). Let $\{\tilde{w}_m\}$ be such a sequence, then

$$J(\tilde{w}_m) \to c, \text{ and } J'(\tilde{w}_m) \to 0. \tag{3.4.9}$$

We proceed now as in [**18**, Lemma 3.6] which is based on ideas contained in [**11**]. By contradiction, suppose that there exists $\eta_0 > 0$, and $m_0 \in \mathbb{N}$ such that for any $\rho > 0$ one has, up to a subsequence,

$$\int_{\{y>\rho\}}\int_\Omega y^{1-2s}|\nabla\tilde{w}_m|^2dxdy>\eta_0,\ \forall m\geq m_0. \tag{3.4.10}$$

Fix $\varepsilon>0$ (to be determined) and let $r>0$ be such that

$$\int_{\{y>r\}}\int_\Omega y^{1-2s}|\nabla\tilde{w}|^2dxdy<\varepsilon.$$

Let $j=\left[\frac{M}{\kappa_s\varepsilon}\right]$ be the integer part with M the constant in (3.4.5) and $I_k=\{y\in\mathbb{R}^+ : r+k\leq y\leq r+k+1\}$, $k=0,1,\dots,j$. Then

$$\sum_{k=0}^{j}\int_{I_k}\int_\Omega y^{1-2s}|\nabla\tilde{w}_m|^2dxdy\leq\int_{\mathscr{C}_\Omega}y^{1-2s}|\nabla\tilde{w}_m|^2dxdy\leq\frac{M}{\kappa_s}<\varepsilon(j+1).$$

Then, there exists $k_0\in\{0,\dots,j\}$ such that, up to a subsequence,

$$\int_{I_{k_0}}\int_\Omega y^{1-2s}|\nabla\tilde{w}_m|^2dxdy\leq\varepsilon,\ \forall m\geq m_0. \tag{3.4.11}$$

We set now a regular cut-off function

$$\chi(y)=\begin{cases}0 & \text{if } y\leq r+k_0,\\ 1 & \text{if } y>r+k_0+1,\end{cases}$$

and we define $v_m(x,y)=\chi(y)\tilde{w}_m(x,y)$. Then, since $v_m(x,0)=0$, it follows that

$$\begin{aligned}|\langle J'(\tilde{w}_m)-J'(v_m),v_m\rangle| &=\kappa_s\int_{\mathscr{C}_\Omega}y^{1-2s}\langle\nabla(\tilde{w}_m-v_m),\nabla v_m\rangle dxdy\\ &=\kappa_s\int_{I_{k_0}}\int_\Omega y^{1-2s}\langle\nabla(\tilde{w}_m-v_m),\nabla v_m\rangle dxdy.\end{aligned}$$

Moreover, by the Cauchy-Schwarz inequality, (3.4.11) and the compact inclusion of the space $H^1\left(I_{k_0}\times\Omega,y^{1-2s}dxdy\right)$ into $L^2\left(I_{k_0}\times\Omega,y^{1-2s}dxdy\right)$, it follows that

$$\begin{aligned}&|\langle J'(\tilde{w}_m)-J'(v_m),v_m\rangle|\\ &\leq\kappa_s\left(\int_{I_{k_0}}\int_\Omega y^{1-2s}|\nabla(\tilde{w}_m-v_m)|^2dxdy\right)^{1/2}\left(\int_{I_{k_0}}\int_\Omega y^{1-2s}|\nabla v_m|^2dxdy\right)^{1/2}\\ &\leq C\kappa_s\varepsilon.\end{aligned}$$

Finally, by (3.4.9),

$$|\langle J'(v_m),v_m\rangle|\leq C\kappa_s\varepsilon+o(1),$$

thus, for m big enough

$$\int_{\{y>r+k_0+1\}}\int_\Omega y^{1-2s}|\nabla w_m|^2dxdy\leq\int_{\mathscr{C}_\Omega}\int_\Omega y^{1-2s}|\nabla v_m|^2dxdy\leq\frac{\langle J'(v_m),v_m\rangle}{\kappa_s}\leq C\varepsilon,$$

which contradicts (3.4.10). Then, the proof of Theorem 3.4.2 is complete. $\square$

Remark 3.4.1. *Note that the proof of Theorem 3.2.3 was done in the first part of the proof of Theorem 3.4.2.*

Now we prove Theorems 3.2.1, 3.2.2.

Proof of Theorem 3.2.2. Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ be a minimizing sequence of $\widetilde{S}(\Sigma_{\mathcal{D}})$ and w its weak limit. By Theorem 3.4.2, $\{w_m\}$ is relatively compact, and consequently the infimum is achieved, or $w \equiv 0$ and

$$y^{1-2s}|\nabla w_n|^2 \to \mu\delta_{x_0}, \text{ and } |u_n|^{2_s^*} \to \nu\delta_{x_0},$$

with $x_0 \in \overline{\Sigma}_{\mathcal{N}}$. Indeed, we can assume, without loss of generality, that $\mu = \widetilde{S}(\Sigma_{\mathcal{D}})$ and $\nu = 1$. With the same notation as in the proof of Theorem 3.4.2, we consider the functions

$$w_{m,r} = w_m\overline{\phi}_r(x,y) \tag{3.4.12}$$

with $\overline{\phi}_r(x,y)$ a smooth cut-off function centered at $x_0 \in \overline{\Sigma}_{\mathcal{N}}$. Clearly, (3.4.12) satisfies (3.4.8). Since $\Sigma_{\mathcal{N}}$ is smooth, for r small enough, the sequence $\{u_{m,r}\} \subset H^s_{\Sigma^r_{\mathcal{D}}}(\Omega_r)$, or equivalently, the sequence $\{w_{m,r}\} \subset \mathcal{X}^s_{\Sigma^r_{\mathcal{D}}}(\mathscr{C}_{\Omega_r})$ thus,

$$\lim_{r\to 0} \frac{\displaystyle\int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla w_{m,r}|^2 dxdy}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} \geq 2^{\frac{-2s}{N}}\kappa_s S(s,N) > \widetilde{S}(\Sigma_{\mathcal{D}}),$$

which contradicts (3.4.8). Then the only possibility is that $\{w_m\}$ is relatively compact, which proves the assertion. □

Proof of Theorem 3.2.1. Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)$ be a minimizing sequence for $S_\lambda(\Omega)$ and w its weak limit. Thus, either $\{w_m\}$ is relatively compact and consequently the infimum is achieved or by Theorem 3.4.2, (3.4.4) holds up to a subsequence. For that sequence we consider the functions $w_{m,r} = w_m\overline{\phi}_r(x,y)$, with $\overline{\phi}_r(x,y)$ a smooth cut-off function centered at $x_0 \in \overline{\Sigma}_{\mathcal{N}}$ as in (3.4.12). On the one hand, $\{w_{m,r}\}$ and its trace $\{u_{m,r}\}$ satisfy

$$\frac{\|w_{m,r}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} - \lambda\|u_{m,r}\|^2_{L^2(\Omega)}}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} \to S_\lambda(\Omega), \quad \text{as} \quad m \to \infty, \tag{3.4.13}$$

for any $r > 0$. On the other, by the definition of $\widetilde{S}(\Sigma_{\mathcal{N}})$ we have

$$\lim_{r\to 0} \frac{\|w_{m,r}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} - \lambda\|u_{m,r}\|^2_{L^2(\Omega)}}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} \geq \widetilde{S}(\Sigma_{\mathcal{N}}),$$

which contradicts (3.4.13) since we are supposing $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$. Hence $\{w_m\}$ is relatively compact. □

Proof of Theorem 3.1.1-(2). By Theorem 3.2.1, it follows inmediatly the existence of a solution of problem (P^c_λ) whenever we have $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{N}})$, which is guaranteed by Proposition 3.4.1 if $0 < \lambda < \lambda_{1,s}$. Also, there exists a solution when $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ by Theorem 3.2.3.

Specifically, by Theorem 3.2.1 and Proposition 3.4.1, if $0<\lambda<\lambda_{1,s}$ there exists a minimizer function $\tilde{w}$ with $\tilde{u}=Tr[\tilde{w}]$ satisfying

$$\|\tilde{w}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}-\lambda\|\tilde{u}\|^2_{L^2(\Omega)}=S_\lambda(\Omega)\|\tilde{u}\|^2_{L^{2^*_s}(\Omega)}.$$

Taking $w=\tilde{w}/\|\tilde{u}\|_{L^{2^*_s}(\Omega)}$ and its trace, $u=\tilde{u}/\|\tilde{u}\|_{L^{2^*_s}(\Omega)}$,

$$\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}-\lambda\|u\|^2_{L^2(\Omega)}=S_\lambda(\Omega). \tag{3.4.14}$$

Thus, w is a minimizer of $S_\lambda(\Omega)$ constrained to the sphere $\|u\|_{L^{2^*_s}(\Omega)}=1$, or equivalently, w is a critical point of the functional Q_λ constrained to $\|u\|^2_{L^{2^*_s}(\Omega)}=1$. Without loss of generality we can assume $w\geq 0$, otherwise we take $|w|$ instead, then thanks to (1.2.1) and (1.2.2), such a critical point is a non-negative solution of equation

$$(-\Delta)^s u-\lambda u=\tau u^{2^*_s-1} \text{ in } \Omega,$$

where $\tau\in\mathbb{R}$ is a Lagrange multiplier. Moreover $\tau=S_\lambda(\Omega)>0$ since $\lambda<\lambda_{1,s}$. Thus, it follows that defining $v=ku$, it is a non-negative solution of the equation in (P^c_λ) for $k=(S_\lambda(\Omega))^{\frac{1}{2^*_s-2}}$. Even more, by the maximum principle, $v>0$ in Ω, proving that it is a solution of (P^c_λ). □

To complete the proof of Theorem 3.1.1 it only remains to prove statement (3) in Theorem 3.1.1. This will be done in the next subsection.

3.4.2. Moving the boundary conditions. Proof of Theorem 3.1.1-(3).

Let us consider the following eigenvalue problem

$$(EP_\alpha)\qquad \left\{\begin{array}{ll} (-\Delta)^s u=\lambda_{1,s}(\alpha)u & \text{in } \Omega\subset\mathbb{R}^N,\\ u=0 & \text{on } \Sigma_{\mathcal{D}}(\alpha),\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \Sigma_{\mathcal{N}}(\alpha),\end{array}\right.$$

with the following hypotheses:

B_1 : $\Omega\subset\mathbb{R}^N$ is a regular bounded domain.
B_2 : $\Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}}(\alpha)$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_{\mathcal{D}}(\alpha)\cup\Sigma_{\mathcal{N}}(\alpha)=\partial\Omega$, $\Sigma_{\mathcal{D}}(\alpha)\cap\Sigma_{\mathcal{N}}(\alpha)=\emptyset$, and the interphase $\Gamma(\alpha)=\Sigma_{\mathcal{D}}(\alpha)\cap\overline{\Sigma}_{\mathcal{N}}(\alpha)$ is a $(N-2)$-dimensional submanifold.
B_3 : $|\Sigma_{\mathcal{D}}(\alpha)|=\alpha$, $\Sigma_{\mathcal{D}}(\alpha_1)\subseteq\Sigma_{\mathcal{D}}(\alpha_2)$ for any $0<\alpha_1\leq\alpha_2<|\partial\Omega|$.

Following [**34**, Lemma 4.1] we have the next result.

LEMMA 3.4.2. *Let u_α be a positive solution of problem (EP_α) and suppose hypotheses B_1-B_3. Then we obtain,*

$$\lambda_{1,s}(\alpha)\to 0, \text{ as } \alpha\to 0.$$

PROOF. By the definition of the fractional operator $(-\Delta)^s$, we have that the eigenvalue $\lambda_{1,s}(\alpha)=\lambda^s_{1,1}(\alpha)$ and, because of [**34**, Lemma 4.1], we have $\lambda_{1,1}(\alpha)\to 0$ as $|\Sigma_{\mathcal{D}}(\alpha)|=\alpha\to 0$. Then the result follows. □

The next proposition is the analogous to [**2**, Proposition 2.1] for our fractional setting.

PROPOSITION 3.4.2. *Let $\Omega\subset\mathbb{R}^N$ be a smooth bounded domain. Given a family $\{\Sigma_{\mathcal{D}}(\alpha):0<\alpha<|\partial\Omega|\}$ satisfying hypotheses B_1-B_3, there exists a positive constant α_0 such that for any $\alpha<\alpha_0$, $\widetilde{S}(\Sigma_{\mathcal{D}}(\alpha))$ is attained.*

Proof. We only have to check that hypotheses of Theorem 3.2.2 are satisfied. To do so, we use the Hölder inequality together with Lemma 3.4.2 as follows. By Hölder's inequality,

$$\begin{aligned}
\widetilde{S}(\Sigma_{\mathcal{D}}(\alpha)) &= \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}(\alpha)}}(\mathscr{C}_\Omega)\\ w\not\equiv 0}} \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)}} \\
&\leq |\Omega|^{\frac{2s}{N}} \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}(\alpha)}}(\mathscr{C}_\Omega)\\ w\not\equiv 0}} \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^{2}(\Omega)}} \\
&= |\Omega|^{\frac{2s}{N}} \lambda_{1,s}(\alpha).
\end{aligned} \tag{3.4.15}$$

Applying Lemma 3.4.2 into (3.4.15), we have that there exists $\alpha_0 > 0$ such that $\widetilde{S}(\Sigma_{\mathcal{D}}(\alpha)) < 2^{\frac{-2s}{N}}\kappa_s S(s,N)$ for any $\alpha < \alpha_0$. Hence, by Theorem 3.2.2 the result follows. □

We complete now the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1-(3). Since $S_\lambda(\Omega) = \widetilde{S}(\Sigma_{\mathcal{D}})$ for $\lambda = 0$, the existence of solution of problem (P_0) is equivalent to the attainability of $\widetilde{S}(\Sigma_{\mathcal{D}})$. Thus, letting α sufficiently small, by Proposition 3.4.2 there exists a minimizer function $\tilde{w}$ with $\tilde{u} = Tr[\tilde{w}]$ satisfying

$$\|\tilde{w}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathscr{C}_\Omega)} = \widetilde{S}(\Sigma_{\mathcal{D}})\|\tilde{u}\|^2_{L^{2^*_s}(\Omega)},$$

and we are done. □

Remark 3.4.2. *By Proposition 3.4.1 and Proposition 3.3.1, if* $0 < \lambda < \lambda_{1,s}$ *then* $S_\lambda(\Omega) < 2^{\frac{-2s}{N}}\kappa_s S(s,N) = \widetilde{S}(\Sigma_{\mathcal{N}})$ *and we are in the hypotheses of Theorem 3.2.1. On the other hand, due to Proposition 3.3.2, for* $0 \leq \lambda < \lambda_{1,s}$*, we obtain the weaker estimate* $0 < S_\lambda(\Omega) \leq \widetilde{S}(\Sigma_{\mathcal{D}}) \leq 2^{\frac{-2s}{N}}\kappa_s S(s,N)$*. Now, the corresponding hypotheses of Theorem 3.2.3 are not fulfilled. Nevertheless, using Proposition 3.4.2 we find* $\widetilde{S}(\Sigma_{\mathcal{D}}) < 2^{\frac{-2s}{N}}\kappa_s S(s,N) = \widetilde{S}(\Sigma_{\mathcal{N}})$ *for* $\alpha = |\Sigma_{\mathcal{D}}|$ *small enough. Therefore, by means the constant* $\widetilde{S}(\Sigma_{\mathcal{D}})$ *we conclude* $0 < S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{N}})$ *for* $0 \leq \lambda < \varepsilon$ *and* $\varepsilon > 0$ *small enough.*

3.5. A nonexistence result: Pohozaev-type identity

This last part deals with a non-existence result relying on a Pohozaev-type identity. Notice that by Theorem 3.1.1-(3) we have the existence of solution of the following critical problem,

$$\left\{\begin{array}{ll}
(-\Delta)^s u = u^{2^*_s-1} & \text{in } \Omega \subset \mathbb{R}^N,\\
u > 0 & \text{in } \Omega,\\
B(u) = 0 & \text{on } \partial\Omega = \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}},
\end{array}\right. \tag{3.5.1}$$

provided $\alpha = |\Sigma_{\mathcal{D}}|$ is small enough, in contrast to the non-existence results for the Dirichlet boundary data case and Ω a star-shaped domain, see Pohozaev [**74**], in the classical setting or [**24**] for the fractional case under the same geometrical hypotheses. Nevertheless, and in spite of Theorem 3.1.1-(3), proceeding in a similar way as in [**67, 61**] we are going to show a Pohozaev-type identity for our fractional mixed Dirichlet-Neumann problems that provides us a non-existence result under appropriate assumptions on the geometry of Ω, $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$.

Let us consider the problem

$$(P_f) \qquad \begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega = \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}}. \end{cases}$$

We have the following result.

PROPOSITION 3.5.1. *Suppose that u is a solution of problem (P_f), $w = E_s[u]$ and f is a continuous function with primitive F. Then the following Pohozaev-type identity holds,*

$$(N-2s)\int_\Omega uf(u)dx - 2N\int_\Omega F(u)dx$$

$$= \kappa_s \int_{\Sigma_{\mathcal{N}}^*} y^{1-2s}|\nabla w|^2 \langle x, \nu\rangle d\sigma(x,y) - \kappa_s \int_{\Sigma_{\mathcal{D}}^*} y^{1-2s}|\nabla w|^2 \langle x, \nu\rangle d\sigma(x,y) \tag{3.5.2}$$

$$-2\int_{\Sigma_N} F(u)\langle x, \nu\rangle d\sigma(x),$$

where ν denotes the outwards normal vector to $\partial\Omega$.

PROOF. Since $w = E_s[u]$ is a solution of problem

$$(P_f^*) \qquad \begin{cases} -div(y^{1-2s}\nabla w) = 0 & \text{in } \mathscr{C}_\Omega, \\ B(w) = 0 & \text{on } \partial_L \mathscr{C}_\Omega, \\ \frac{\partial w}{\partial \nu^s} = f(u) & \text{in } \Omega, \end{cases}$$

multiplying the equation of (P_f^*) by $\varphi(x,y)$ and integrating by parts we get

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\nabla w \nabla\varphi dxdy = \int_\Omega \varphi(x,0)f(u)dx + \kappa_s \int_{\Sigma_{\mathcal{D}}^*} \varphi y^{1-2s}\langle \nabla w, \nu^*\rangle d\sigma(x,y).$$

With ν^* the outwards normal vector to $\partial_L\mathscr{C}_\Omega$. We take $\varphi(x,y) = \langle (x,y), \nabla w\rangle$ and note that $\langle \nabla w, \nu^*\rangle = |\nabla w|$ on $\Sigma_{\mathcal{D}}^*$, as well that, by construction, the outwards normal vector ν^* to the lateral boundary $\partial_L\mathscr{C}_\Omega$ verifies $\nu^* = (\nu, 0)$ with ν the outwards normal vector to $\partial\Omega$. Then, we find,

$$\frac{2s-N}{2}\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy + \frac{1}{2}\kappa_s \int_{\partial_L\mathscr{C}_\Omega} y^{1-2s}|\nabla w|^2\langle x, \nu\rangle d\sigma(x,y) =$$
$$\int_{\Sigma_{\mathcal{N}}} F(u)\langle x, \nu\rangle d\sigma(x) - N\int_\Omega F(u)dx + \kappa_s \int_{\Sigma_{\mathcal{D}}^*} y^{1-2s}|\nabla w|^2\langle x, \nu\rangle d\sigma(x,y),$$

which proves (3.5.2). □

As a direct consequence of Proposition 3.5.1 we obtain a non-existence result for problem P_f.

THEOREM 3.5.1. *Assume the hypotheses of Proposition 3.5.1 and suppose there exists $x_0 \in \Omega$ such that $\langle x - x_0, \nu\rangle = 0$ on $\Sigma_{\mathcal{N}}$ and $\langle x - x_0, \nu\rangle > 0$ on $\Sigma_{\mathcal{D}}$. If f and F satisfy the inequality $(N-2s)tf(t) - 2NF(t) \geq 0$, then problem (P_f) has no solution.*

This result highlights the difference between a mixed boundary condition problem and a Dirichlet one as well as the relevance of the geometry of Ω and the decomposition of $\partial\Omega$ into $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ in the existence issues.
As an example, let us consider the critical power problem (3.5.1) with Ω defined as follows. Given A_α a smooth submanifold of the unit sphere $\mathbb{S}^{N-1}$ such that $|A_\alpha| = \alpha$, we set $\Omega = \{tx : x \in A_\alpha, 0 < t < R\}$, $\Sigma_{\mathcal{D}} = \{x \in \overline{\Omega} : |x| = R\}$ and $\Sigma_{\mathcal{N}} = \partial\Omega \backslash \Sigma_{\mathcal{D}}$. We consider a smooth perturbation $\widetilde{\Omega}$ where the vertex $x_0 = \overline{0}$ and the corners of Ω are regularized, such that $|\widetilde{\Omega} \backslash \Omega|$ is small enough. Set $\widetilde{\Sigma}_{\mathcal{D}} = \Sigma_{\mathcal{D}}$ and $\widetilde{\Sigma}_{\mathcal{N}} = \partial\widetilde{\Omega} \backslash \widetilde{\Sigma}_{\mathcal{D}}$. Then, $\langle x, \nu \rangle = 0$ on $\widetilde{\Sigma}_{\mathcal{N}} \backslash T_\rho$ and $\langle x, \nu \rangle \neq 0$ on $\widetilde{\Sigma}_{\mathcal{N},\rho} = \widetilde{\Sigma}_{\mathcal{N}} \cap T_\rho$ with $T_\rho = B_\rho(0) \cup \{x \in \mathbb{R}^N : R - \rho < |x| < R\}$ and some $\rho > 0$ small enough, as well as $\langle x, \nu \rangle > 0$ on $\widetilde{\Sigma}_{\mathcal{D}}$. Since we can approximate the cone Ω arbitrarily by means of $\widetilde{\Omega}$, we can let ρ be sufficiently small in order to obtain a contradiction with the Pohozaev identity, namely

$$\frac{N-2s}{N} \int_{\widetilde{\Sigma}_{\mathcal{N},\rho}} |u|^{2_s^*} \langle x, \nu \rangle d\sigma = \kappa_s \int_{\widetilde{\Sigma}^*_{\mathcal{N},\rho}} y^{1-2s} |\nabla w|^2 \langle x, \nu \rangle d\sigma + R\kappa_s \int_{\widetilde{\Sigma}^*_{\mathcal{D}}} y^{1-2s} |\nabla w|^2 d\sigma. \tag{3.5.3}$$

Thus, no solution of the problem (3.5.1) exists on $\widetilde{\Omega}$ for $\rho > 0$ small snough.

Remark 3.5.1. *If we move the boundary conditions in the example above, letting* $|\Sigma_{\mathcal{D}}| \to 0$*, by means of Theorem 3.1.1-*(3) *we get the existence of solution of problem* (3.5.1) *on the perturbed cone* $\widetilde{\Omega}$*. This is not in contradiction with the previous arguments, because by this procedure, points that belonged to the Dirichlet boundary part for which we had* $\langle x, \nu \rangle > 0$*, start to contribute to the integral involving the Neumann part of the boundary in* (3.5.3)*, and hence Theorem 3.1.1-*(3) *and Theorem 3.5.1 agree.*

Part 3

Critical Problems involving inverse operators and Dirichlet Boundary data

CHAPTER 4

Existence of positive solutions for a Brezis-Nirenberg–type problem involving an inverse operator

Throughout this chapter we focus on the study of existence of positive solutions for a problem related to a fourth-order differential equation involving a nonlinear term depending on a second order differential operator,

$$(-\Delta)^2 u = \lambda u + (-\Delta)|u|^{p-1}u,$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N > 6$, and assuming homogeneous Navier boundary conditions. In particular, we study a second order equation involving a nonlocal term of the form,

$$-\Delta u = \lambda(-\Delta)^{-1}u + |u|^{p-1}u,$$

under Dirichlet boundary conditions and we prove the existence of positive solutions depending on the positive real parameter $\lambda > 0$, up to the critical value of the exponent p, i.e., when $1 < p \leq 2^* - 1$, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. When $p = 2^* - 1$, this equivalence leads us to a similar problem to the classical Brezis-Nirenberg problem, cf. [**26**], but, in our particular case, the linear term is a nonlocal term. The effect that this nonlocal term has on the equation changes the dimensions for which the classical technique based on the minimizers of the Sobolev constant ensures the existence of solution, going from dimensions $N \geq 4$ in the classical Brezis-Nirenberg problem, to dimensions $N > 6$ for this nonlocal problem.

4.1. Introduction

Our main aim along this chapter is to study the existence of positive solutions of a problem derived from the following fourth-order equation under homogeneous Navier boundary conditions,

$$(P_\lambda^2) \qquad \left\{ \begin{array}{rll} (-\Delta)^2 u & = \lambda u + (-\Delta)|u|^{p-1}u & \text{in} \quad \Omega \subset \mathbb{R}^N, \\ u & = 0 & \text{on} \quad \partial\Omega, \\ -\Delta u & = 0 & \text{on} \quad \partial\Omega, \end{array} \right.$$

where λ is a positive real parameter and Ω is a smooth bounded domain of $\mathbb{R}^N$, with $N > 6$. This important fact on the dimension will be under review along this chapter. In particular, positive solutions of (P_λ^2) can be seen as positive steady-state solutions of the *fourth-order parabolic Cahn–Hilliard type-equation*,

$$\frac{\partial u}{\partial t} + (-\Delta)^2 u = \lambda u + (-\Delta)|u|^{p-1}u, \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$

assuming bounded smooth initial data $u(x,0) = u_0(x)$. The latter equation has been previously studied in [**3, 6, 5**] for bounded domains or the whole $\mathbb{R}^N$ but considering exponents

p in the subcritical range $1 < p < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ is the critical exponent of the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. The results contained in this chapter extend the former range and deal with exponents $1 < p \leq 2^* - 1$, covering the critical exponent case. Let us recall that, because of the Sobolev Embedding Theorem, we have the compact embedding,

$$H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega), \tag{4.1.1}$$

for $2 \leq p+1 < 2^*$, being a continuous embedding up to the critical exponent $p = 2^* - 1$. Moreover, given $u \in H_0^1(\Omega)$, because of the Sobolev inequality, there exist a positive constant $C = C(N,p)$ such that

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}, \tag{4.1.2}$$

for $2 \leq p+1 \leq 2^*$. Note that here, for the fourth-order elliptic problem (P_λ^2), the Sobolev's critical exponent we are using is $2^* = \frac{2N}{N-2}$, because this operator has the representation,

$$(-\Delta)^2 u - (-\Delta)|u|^{p-1}u = (-\Delta)((-\Delta)u - |u|^{p-1}u),$$

so that, the necessary embedding features are governed by a standard second-order equation,

$$-\Delta u = |u|^{p-1}u.$$

This is different from the usual critical problems with a bi-Laplacian operator of the form,

$$(-\Delta)^2 u = \lambda u + |u|^{p-1}u,$$

analyzed by Gazzola-Grunau-Sweers [**58**], where the Sobolev's critical exponent considered is $p_S = \frac{2N}{N-4}$.

On the other hand, we also observe that (P_λ^2) is not a variational problem. Nonetheless, applying $(-\Delta)^{-1}$ to the equation of (P_λ^2), we obtain the following nonlocal elliptic Dirichlet problem,

$$(P_\lambda) \qquad \left\{ \begin{array}{ll} -\Delta u = \lambda(-\Delta)^{-1}u + |u|^{p-1}u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

which is a variational problem with the following associated Euler-Lagrange functional,

$$\mathcal{F}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u(-\Delta)^{-1}u\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}dx, \tag{4.1.3}$$

so that solutions of (P_λ) can be obtained as critical points of the Fréchet-differentiable functional $\mathcal{F}_\lambda$ defined by (4.1.3). Here, as customary $(-\Delta)^{-1}u = v$, if

$$-\Delta v = u \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega.$$

Note that $(-\Delta)^{-1}$ is a positive linear compact integral operator from $L^2(\Omega)$ into itself, which is well defined thanks to the Spectral Theorem. Next, we recall the following well-known facts about *polyharmonic* operators of order $2m$ ($m \geq 1$ an integer number) in smooth domains Ω. The Navier boundary conditions for the operator $(-\Delta)^m$ are defined as

$$u = \Delta u = \Delta^2 u = \ldots = \Delta^{k-1}u = 0, \quad \text{on } \partial\Omega.$$

Clearly, the operator $(-\Delta)^m$ is the m-th power of the classical Dirichlet Laplacian in the sense of the spectral theory and it can be defined as the operator whose action on a function u is given by

$$\langle(-\Delta)^m u, u\rangle = \sum_{j\geq 1} \lambda_j^m |\langle u_1, \varphi_j\rangle|^2,$$

where (φ_i, λ_i) are the eigenfunctions and eigenvalues of the Laplace operator $(-\Delta)$ with homogeneous Dirichlet boundary data. Thus, the operator $(-\Delta)^m$ is well defined in the space of functions that vanish on the boundary $\partial\Omega$,

$$H_0^m(\Omega) = \left\{ u = \sum_{j=1}^{\infty} a_j \varphi_j \in L^2(\Omega) : \ ||u||_{H_0^m(\Omega)} = \left(\sum_{j=1}^{\infty} a_j^2 \lambda_j^m \right)^{\frac{1}{2}} < \infty \right\}.$$

Since the above definition allows us to integrate by parts, a natural definition of energy solution for problem (P_λ) is given by critical points of the functional $\mathcal{F}_\lambda$ defined by (4.1.3). Moreover, we can rewrite the functional (4.1.3) as,

$$\mathcal{F}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |(-\Delta)^{-1/2} u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

Additionally, we have a connection between problem (P_λ^2) and a second order elliptic system through problem (P_λ). In particular, taking $w := (-\Delta)^{-1}u$, problem (P_λ) provides us with the system,

$$\text{(4.1.4)} \qquad \left\{ \begin{array}{l} -\Delta u = \lambda w + |u|^{p-1}u, \\ -\Delta w = u, \end{array} \right. \quad \text{in} \quad \Omega, \quad (u,w) = (0,0) \quad \text{on} \quad \partial\Omega,$$

which gives a different perspective to the problem in hand. In fact, we shall obtain the main results of this chapter following both perspectives with respect to the nonlocal equation (P_λ) and the provided by considering a second order elliptic system. Moreover, in order to obtain a variational system from problem (P_λ), and since $\lambda > 0$, we take $v := \sqrt{\lambda} w$ in (4.1.4) and we obtain the variational system

$$(S_\lambda) \qquad \left\{ \begin{array}{l} -\Delta u = \sqrt{\lambda} v + |u|^{p-1}u, \\ -\Delta v = \sqrt{\lambda} u, \end{array} \right. \quad \text{in} \quad \Omega, \quad (u,v) = (0,0) \quad \text{in} \quad \partial\Omega,$$

whose associated Euler-Lagrange functional is

$$\text{(4.1.5)} \qquad \mathcal{J}_\lambda(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \sqrt{\lambda}\int_\Omega uv dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

REMARK 4.1.1. *Because of the Maximum Principle, given u a positive solution to (P_λ), and setting $v = \sqrt{\lambda}(-\Delta)^{-1}u$, it follows that $v > 0$ thus, the pair $(u, v) = (u, \sqrt{\lambda}(-\Delta)^{-1}u)$ is a positive solution to (S_λ) and vice versa, given (u, v) a positive solution to (S_λ) it is immediate that u is a positive solution to (P_λ).*

As we commented in the introductory part of this PhD Thesis dissertation, when one considers the critical exponent case, $p = 2^* - 1$, problem (P_λ) can be seen as a linear perturbation of the critical problem,

$$\text{(4.1.6)} \qquad \left\{ \begin{array}{ll} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

for which, after applying the well-known result of Pohozaev, [**75**], one can prove the nonexistence of positive solutions under the star-shapeness assumption on the domain Ω. Moreover, the classical Brezis-Nirenberg problem,

$$\text{(4.1.7)} \qquad \left\{ \begin{array}{ll} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

can be seen as well as a linear perturbation of problem (4.1.6). In his pioneering paper, [**26**], Brezis and Nirenberg proved that, for $N \geq 4$, there exists a positive solution to (4.1.7) if and only if the parameter λ belongs to the interval $(0, \lambda_1)$, being λ_1 the first eigenvalue for the Laplacian under homogeneous Dirichlet boundary conditions. Note that, in our situation, the nonlocal term $\sqrt{\lambda} v = \lambda(-\Delta)^{-1} u$ plays actually the role of λu in (4.1.7). This important fact is under analysis in Section 4.2.

Main results. We prove the existence of positive solutions to problem (P_λ) depending on the positive parameter λ. To do so, we will first show the interval of the parameter λ for which there is the possibility of having positive solutions. Next, applying the Mountain Pass Theorem, we show that for the range $2 < p + 1 \leq 2^*$ there actually exists at least a positive solution to problem (P_λ) provided

$$0 < \lambda < \lambda_{1,2},$$

where $\lambda_{1,2}$ is the first eigenvalue of the operator $(-\Delta)^2$ under homogeneous Navier boundary conditions, i.e. $\lambda_{1,2} = \lambda_1^2$ with λ_1 being the first eigenvalue for the Laplacian under homogeneous Dirichlet boundary conditions. If $2 < p + 1 < 2^*$ one might apply the Mountain Pass Theorem directly since, as we will show, our problem possesses the mountain pass geometry and, thanks to the compact embedding (4.1.1), the Palais-Smale condition is satisfied for the functional $\mathcal{F}_\lambda$ (see details below in Section 4.2). On the other hand, at the critical exponent 2^*, the compactness of the Sobolev embedding is lost and check whether the PS condition is satisfied becomes a delicate issue to solve. To overcome this lack of compactness we apply a concentration-compactness argument based on the Concentration-Compactness Principle due to P. L. Lions, [**65**], which allows us to prove the required Palais-Smale condition for $N > 6$. We prove the results for problem (P_λ) in Section 4.2 and using similar ideas, for system (S_λ) in Section 4.3.
Now we state the main results of this chapter.

THEOREM 4.1.1. *Assume $1 < p < 2^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,2})$ there exists at least a positive solution u to problem (P_λ).*

THEOREM 4.1.2. *Assume $p = 2^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,2})$, there exists at least a positive solution u to problem (P_λ) provided $N > 6$.*

As we have commented before, even though our problem (P_λ) is a nonlocal but also a linear perturbation of the problem (4.1.6), Theorem 4.1.2 addresses dimensions $N > 6$, in contrast to the existence result of Brezis and Nirenberg about the linear perturbation (4.1.7), that covers the wider range $N \geq 4$. As we will see trhoughout this chapter, the nonlocal term $\lambda(-\Delta)^{-1} u$ has an important effect on the dimensions for which the classical Brezis-Nirenberg technique based on the minimizers of the Sobolev constant still works.
Finally, although the equivalence between the system (S_λ) and the nonlocal problem (P_λ) provides us with existence results for the system (S_λ) by means of Theorem 4.1.1 and Theorem 4.1.2, we prove independently the following.

THEOREM 4.1.3. *Assume $1 < p < 2^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,2})$, there exists at least a positive solution (u, v) to system (S_λ).*

THEOREM 4.1.4. *Assume $p = 2^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,2})$, there exists at least positive solution (u, v) to system (S_λ) provided $N > 6$.*

In the last section of the chapter we extend our study and prove, under similar hypotheses above, that there exists at least a positive solution to the problem

$$\begin{cases} -\Delta u = (-\Delta)^{-m}\lambda u + |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Due to the lack of a comparison principle for a higher order equations, to obtain the existence results we can not address this problem directly, and we need to use a similar correspondence to the one performed above for the problem (P_λ^2), now with an elliptic system of $m+1$ equations.

4.2. Existence of positive solutions for the nonlocal problem (P_λ)

In this section we carry out the proof of Theorem 4.1.1 and Theorem 4.1.2. This is done through the equivalence between equation (P_λ^2) and equation (P_λ), hence, we will develop the work in the variational setting inherited from this consideration. First, we establish a condition on the range of values of the parameter λ necessary for the existence of positive solutions to equation (P_λ^2). Let us consider the following generalized eigenvalue problem associated to (P_λ),

$$\begin{cases} -\Delta u = \lambda(-\Delta)^{-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2.1}$$

Then, we find that for the first eigenfunction ϕ_1 associated with the first eigenvalue λ_1^* in (4.2.1),

$$\int_\Omega |\nabla\phi_1|^2 dx = \lambda_1^* \int_\Omega |(-\Delta)^{-1/2}\phi_1|^2 dx, \quad \text{with} \quad \phi_1 \in H_0^1(\Omega),$$

and, hence,

$$\lambda_1^* = \inf_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega |(-\Delta)^{-1/2}\phi|^2 dx}. \tag{4.2.2}$$

On the other hand, it is clear that substituting the first eigenfunction of the Laplace operator under homogeneous Dirichlet boundary conditions, φ_1, into (4.2.1), it follows that $\lambda_1^* = \lambda_1^2$. Thus, by the very definition of the powers of the Laplace operator, λ_1^* coincides with the first eigenvalue $\lambda_{1,2}$ of the operator $(-\Delta)^2$ under homogeneous Navier boundary conditions as well as the first eigenfunction ϕ_1 of (4.2.1) coincides with the first eigenfunction of the Laplace operator under homogeneous Dirichlet boundary conditions. Now, we prove the following.

Lemma 4.2.1. *Problem (P_λ) does not possess a positive solution when*

$$\lambda \geq \lambda_{1,2}.$$

Proof. Assume that u is a positive solution to (P_λ) and let φ_1 be a positive first eigenfunction of the Laplacian operator in Ω under homogeneous Dirichlet boundary conditions. Taking φ_1 as a test function for the equation of (P_λ) we obtain,

$$\begin{aligned} \int_\Omega \varphi_1(-\Delta)u dx &= \lambda \int_\Omega \varphi_1(-\Delta)^{-1}u dx + \int_\Omega |u|^{p-1}u\varphi_1 dx \\ &> \lambda \int_\Omega \varphi_1(-\Delta)^{-1}u dx. \end{aligned} \tag{4.2.3}$$

Thus, integrating by parts both sides of (4.2.3),

$$\lambda_1 \int_\Omega u\varphi_1 dx > \lambda \int_\Omega u(-\Delta)^{-1}\varphi_1 dx = \frac{\lambda}{\lambda_1}\int_\Omega u\varphi_1 dx.$$

Hence, $\lambda < \lambda_1^2 = \lambda_{1,2}$. □

LEMMA 4.2.2. *The functional $\mathcal{F}_\lambda$ denoted by* (4.1.3) *has the Mountain Pass (MP) geometry.*

PROOF. Without loss of generality, let us take $g \in H_0^1(\Omega)$ such that $\|g\|_{L^{p+1}(\Omega)} = 1$. Then, taking a real number $t > 0$ and applying the Sobolev inequality (4.1.2) together with (4.2.2), we find that,

$$\begin{aligned}
\mathcal{F}_\lambda(tg) &= \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx - \frac{t^2\lambda}{2}\int_\Omega |(-\Delta)^{-\frac{1}{2}} g|^2 dx - \frac{t^{p+1}}{p+1} \\
&\geq \frac{t^2}{2}\left(1 - \frac{\lambda}{\lambda_{1,2}}\right)\int_\Omega |\nabla g|^2 dx - \frac{t^{p+1}}{p+1} \\
&\geq \left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{1,2}}\right) t^2 - \frac{C}{(p+1)} t^{p+1}\right)\int_\Omega |\nabla g|^2 dx \\
&> 0
\end{aligned}$$

for t small enough, i.e.

$$0 < t^{p-1} < \frac{p+1}{2C}\left(1 - \frac{\lambda}{\lambda_{1,2}}\right).$$

Thus, the functional $\mathcal{F}_\lambda$ has a local minimum at $u = 0$, i.e.

$$\mathcal{F}_\lambda(tg) > \mathcal{F}_\lambda(0) = 0,$$

for any $g \in H_0^1(\Omega)$ provided $t > 0$ is small enough. Also, it is clear that,

$$\begin{aligned}
\mathcal{F}_\lambda(tg) &= \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx - \frac{\lambda t^2}{2}\int_\Omega |(-\Delta)^{-1/2} g|^2 dx - \frac{t^{p+1}}{p+1} \\
&\leq \frac{t^2}{2}\|g\|^2_{H_0^1(\Omega)} - \frac{t^{p+1}}{p+1}.
\end{aligned}$$

Then,

$$\mathcal{F}_\lambda(tg) \to -\infty, \quad \text{as} \quad t \to \infty,$$

and thus, there exists $\hat{u} \in H_0^1(\Omega)$ such that $\mathcal{F}_\lambda(\hat{u}) < 0$.

□

Now we turn our attention to the so-called Palais-Smale condition.

DEFINITION 4.2.1. *Let V be a Banach space. We say that a sequence $\{u_n\} \subset V$ is a PS sequence for a functional $\mathfrak{F}$ iff*

$$\mathfrak{F}(u_n) \quad \text{is bounded and} \quad \mathfrak{F}'(u_n) \to 0 \quad \text{in } V' \quad \text{as} \quad n \to \infty, \tag{4.2.4}$$

where V' is the dual space of V. Moreover, we say that a PS sequence $\{u_n\} \subset V$ satisfies a PS condition iff

$$\{u_n\} \quad \text{has a convergent subsequence.} \tag{4.2.5}$$

In particular, given a PS sequence $\{u_n\} \subset V$ such that $\mathfrak{F}(u_n) \to c$, if (4.2.5) is satisfied, we will say that the PS sequence satisfies a PS condition at level c for the functional $\mathfrak{F}$. Moreover, we say that the functional $\mathfrak{F}$ satisfies the PS condition at level c if every PS sequence at level c for $\mathfrak{F}$ possesses a convergent subsequence in V.
For our problem, in the subcritical range the PS condition is always satisfied at any level c because of the compact Sobolev embedding (4.1.1). However, at the critical exponent 2^* the problem is further complicated because of the lack of compactness in the Sobolev embedding. We will overcome this issue applying a concentration-compactness argument based on the Concentration-Compactness Principle developed by P. L. Lions, [**65**], proving that the functional $\mathcal{F}_\lambda$ satisfies the PS condition for levels c below a certain critical value c^* (to be determined).
In a first step, we will prove the results for the functional $\mathcal{F}_\lambda$ containing the nonlocal term. To this end, we will obtain estimates for the nonlocal term that, as shown below, play an important role in the proof of Theorem 4.1.2. In the next section we will work with the functional $\mathcal{J}_\lambda$ associated with the cooperative system (S_λ), avoiding the nonlocal term, and arriving at the same results.

LEMMA 4.2.3. *Let $\{u_n\}$ be a PS sequence at level c for the functional $\mathcal{F}_\lambda$, i.e.*

$$\mathcal{F}_\lambda(u_n) \to c, \quad \mathcal{F}'_\lambda(u_n) \to 0, \quad \textit{as} \quad n \to \infty.$$

Then,

$$\{u_n\} \quad \textit{is bounded in} \quad H_0^1(\Omega).$$

PROOF. Since $\mathcal{F}'_\lambda(u_n) \to 0$ in $\left(H_0^1(\Omega)\right)'$, in particular we have $\left\langle \mathcal{F}'_\lambda(u_n) | \frac{u_n}{\|u_n\|_{H_0^1(\Omega)}} \right\rangle \to 0.$
Thus, for any $\varepsilon > 0$ there exists a subsequence, denoted again by $\{u_n\}$, such that,

$$\int_\Omega |\nabla u_n|^2 dx - \lambda \int_\Omega |(-\Delta)^{-\frac{1}{2}} u_n|^2 dx - \int_\Omega |u_n|^{p+1} dx = \|u_n\|_{H_0^1(\Omega)} \cdot o(1).$$

Moreover, since $\mathcal{F}_\lambda(u_n) \to c$,

$$\frac{1}{2} \int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{2} \int_\Omega |(-\Delta)^{-\frac{1}{2}} u_n|^2 dx - \frac{1}{p+1} \int_\Omega |u_n|^{p+1} dx = c + o(1),$$

for n big enough. Therefore, for a positive constant μ (to be determined below) we find that

$$\mathcal{F}_\lambda(u_n) - \mu \left\langle \mathcal{F}'_\lambda(u_n) | \frac{u_n}{\|u_n\|_{H_0^1(\Omega)}} \right\rangle = c + \|u_n\|_{H_0^1(\Omega)} \cdot o(1).$$

That is,

$$\left(\frac{1}{2} - \mu\right) \int_\Omega |\nabla u_n|^2 dx - \left(\frac{1}{2} - \mu\right) \lambda \int_\Omega |(-\Delta)^{-\frac{1}{2}} u_n|^2 dx - \left(\frac{1}{p+1} - \mu\right) \int_\Omega |u_n|^{p+1} dx$$
$$= c + \|u_n\|_{H_0^1(\Omega)} \cdot o(1).$$

Hence, taking μ such that $\frac{1}{p+1} < \mu < \frac{1}{2}$,

$$\left(\frac{1}{2} - \mu\right) \int_\Omega |\nabla u_n|^2 dx - \left(\frac{1}{2} - \mu\right) \lambda \int_\Omega |(-\Delta)^{-\frac{1}{2}} u_n|^2 dx \leq c + \|u_n\|_{H_0^1(\Omega)} \cdot o(1),$$

and using (4.2.2),

$$\left(\frac{1}{2}-\mu\right)\left(1-\frac{\lambda}{\lambda_{1,2}}\right)\int_\Omega |\nabla u_n|^2dx \le \left(\frac{1}{2}-\mu\right)\int_\Omega |\nabla u_n|^2dx - \left(\frac{1}{2}-\mu\right)\lambda\int_\Omega |(-\Delta)^{-\frac{1}{2}}u_n|^2dx$$
$$\le c+\|u_n\|_{H_0^1(\Omega)}\cdot o(1).$$

From here, we conclude

$$\left(\frac{1}{2}-\mu\right)\left(1-\frac{\lambda}{\lambda_{1,2}}\right)\|u_n\|^2_{H_0^1(\Omega)} \le c+\|u_n\|_{H_0^1(\Omega)}\cdot o(1).$$

Since $0<\lambda<\lambda_{1,2}$, it follows that $\left(\frac{1}{2}-\mu\right)\left(1-\frac{\lambda}{\lambda_{1,2}}\right)>0$ and, thus, because of the former inequality we conclude that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. $\square$

Proof of Theorem 4.1.1. Let us consider the subcritical case $1<p<2^*-1$. Given a PS sequence $\{u_n\}\subset H_0^1(\Omega)$ at level c, by Lemma 4.2.3 and the Rellich-Kondrachov Theorem the PS condition is satisfied. Hence, the functional $\mathcal{F}_\lambda$ satisfies the PS condition. Moreover, by Lemma 4.2.2 the functional $\mathcal{F}_\lambda$ possesses the MP geometry. Therefore, the hypotheses of the Mountain Pass Theorem are fulfilled and we conclude that the functional $\mathcal{F}_\lambda$ possesses a critical point $u\in H_0^1(\Omega)$. Moreover, if we define the set of paths

$$\Gamma := \{\gamma\in\mathcal{C}([0,1],H_0^1(\Omega))\,;\,\gamma(0)=0,\ \gamma(1)=\hat{u}\},$$

with $\hat{u}$ given as in the proof of Lemma 4.2.2, then,

$$\mathcal{F}_\lambda(u)=c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{F}_\lambda(\gamma(t)).$$

To show that $u>0$, let us consider the functional,

$$\mathcal{F}_\lambda^+(u)=\mathcal{F}_\lambda(u^+),$$

where $u^+=\max\{u,0\}$. Repeating with minor changes the arguments carried out above, one readily shows that what was proved for the functional $\mathcal{F}_\lambda$ still holds for the functional $\mathcal{F}_\lambda^+$. Therefore, $u\ge 0$ and by the Maximum Principle, $u>0$. Then, the proof of existence of positive solutions to problem (P_λ) is completed. $\square$

Remark 4.2.1. *Assuming that $\partial\Omega$ is a $\mathcal{C}^2$ manifold, by standard elliptic regularity theory,* [**47**, Sec. 8.3, Theorem 1]*, it follows that $u\in H_0^1(\Omega)\cap H^2(\Omega)$ and thus, u is a positive weak solution to problem (P_λ^2).*

4.2.1. Concentration-Compactness for the nonlocal problem (P_λ).

In this subsection we focus on the critical exponent case, $p=2^*-1$, and our aim is to prove the PS condition for the functional $\mathcal{F}_\lambda$. We carry out this task by means of a concentration-compactness argument based on the following.

Lemma 4.2.4 (P. L. Lions,[**65**]). *Let $\{u_n\}$ be a weakly convergent sequence to a function u in $H_0^1(\Omega)$. Let μ, and ν be two nonnegative measures such that*

$$|\nabla u_n|^2\to\mu \quad \textit{and} \quad |u_n|^{2^*}\to\nu \quad \textit{as } n\to\infty.$$

Then, there exist a countable set I of points $\{x_j\}_{j\in I}\subset\overline{\Omega}$ and some positive numbers μ_j, and ν_j such that

$$
\begin{aligned}
|\nabla u_n|^2 &\rightharpoonup \mu = |\nabla u_0|^2 + \sum_{j\in I}\mu_j\delta_{x_j},\\
|u_n|^{2^*} &\rightharpoonup \nu = |u_0|^{2^*} + \sum_{j\in I}\nu_j\delta_{x_j},
\end{aligned}
\tag{4.2.6}
$$

where δ_{x_j} is the Dirac's delta centered at x_j and satisfying

$$
\mu_j \geq S_N \nu_j^{2/2^*}. \tag{4.2.7}
$$

Lemma 4.2.5. *Assume $p=2^*-1$. Then, the functional $\mathcal{F}_\lambda$ satisfies the Palais-Smale condition for any level c such that,*

$$
c < c^* = \frac{1}{N}S_N^{N/2}.
$$

Proof. Although the proof is rather standard we include the details for the sake of completeness. Let $\{u_n\}\subset H_0^1(\Omega)$ be a PS sequence of level $c<c^*$ for the functional $\mathcal{F}_\lambda$. Thanks to Lemma 4.2.3, the sequence $\{u_n\}$ is uniformly bounded and, as a consequence, we can assume that, up to a subsequence,

$$
\begin{aligned}
&u_n \rightharpoonup u_0 \quad \text{weakly in} \quad H_0^1(\Omega),\\
&u_n \to u_0 \quad \text{strongly in} \quad L^q(\Omega), 1\leq q<2^*,\\
&u_n \to u_0 \quad \text{a.e. in} \quad \Omega.
\end{aligned}
\tag{4.2.8}
$$

Next, for $j\in I$ and $\varepsilon>0$, let $\varphi_{j,\varepsilon}\in\mathcal{C}_0^\infty(\Omega)$ be a cut-off function such that,

$$
\varphi_{j,\varepsilon}=1 \quad \text{in} \quad B_\varepsilon(x_j), \quad \varphi_{j,\varepsilon}=0 \quad \text{in} \quad B_{2\varepsilon}^c(x_j) \quad \text{and} \quad |\nabla\varphi_{j,\varepsilon}|\leq\frac{2}{\varepsilon}, \tag{4.2.9}
$$

where $B_r(x_j)$ is the ball of radius $r>0$, centered at a point $x_j\in\overline{\Omega}$. Thus, using $\varphi_{j,\varepsilon}u_n$ as a test function we find that,

$$
\begin{aligned}
\langle \mathcal{F}_\lambda'(u_n)|\varphi_{j,\varepsilon}u_n\rangle &= \int_\Omega \nabla u_n\cdot\nabla(\varphi_{j,\varepsilon}u_n)dx - \lambda\int_\Omega \varphi_{j,\varepsilon}u_n(-\Delta)^{-1}u_n dx - \int_\Omega \varphi_{j,\varepsilon}|u_n|^{2^*}dx\\
&= \int_\Omega \varphi_{j,\varepsilon}|\nabla u_n|^2 dx - \int_\Omega \varphi_{j,\varepsilon}|u_n|^{2^*}dx\\
&+ \int_\Omega u_n\nabla u_n\cdot\nabla\varphi_{j,\varepsilon}dx - \lambda\int_\Omega\varphi_{j,\varepsilon}u_n(-\Delta)^{-1}u_n dx.
\end{aligned}
$$

Moreover, due to (4.2.6) and (4.2.8),

$$
\lim_{n\to\infty}\langle \mathcal{F}_\lambda'(u_n)|\varphi_{j,\varepsilon}u_n\rangle = \int_\Omega\varphi_{j,\varepsilon}d\mu - \int_\Omega\varphi_{j,\varepsilon}d\nu - \lambda\int_\Omega\varphi_{j,\varepsilon}u_0(-\Delta)^{-1}u_0dx + \int_\Omega u_0\nabla u_0\cdot\nabla\varphi_{j,\varepsilon}dx.
$$

By construction,

$$
\lim_{\varepsilon\to 0}\left[-\lambda\int_\Omega\varphi_{j,\varepsilon}u_0(-\Delta)^{-1}u_0dx + \int_\Omega u_0\nabla u_0\cdot\nabla\varphi_{j,\varepsilon}dx\right]=0.
$$

Then, as $\mathcal{F}_\lambda'(u_n) \to 0$ in $\left(H_0^1(\Omega)\right)'$, we obtain that,

$$\lim_{\varepsilon\to 0}\left(\int_\Omega \varphi_{j,\varepsilon} d\mu - \int_\Omega \varphi_{j,\varepsilon} d\nu\right) = \mu_j - \nu_j = 0,$$

and we conclude,

$$\nu_j = \mu_j. \tag{4.2.10}$$

Finally, we have two options either the PS sequence has a convergent subsequence or it concentrates around some of the points x_j. In other words, $\nu_j = \mu_j = 0$, or there exists some $\nu_j > 0$ such that, by (4.2.7) and (4.2.10), $\nu_j \geq S_N^{N/2}$. In case of having concentration, we find that

$$\begin{aligned} c = \lim_{n\to\infty} \mathcal{F}_\lambda(u_n) &= \lim_{n\to\infty} \mathcal{F}_\lambda(u_n) - \frac{1}{2}\langle \mathcal{F}_\lambda'(u_n)|u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega |u_0|^{2^*} dx + \left(\frac{1}{2} - \frac{1}{2^*}\right)\nu_j \\ &\geq \frac{1}{N} S_N^{N/2} = c^*, \end{aligned}$$

in contradiction with the hypotheses $c < c^*$. Therefore, the PS sequence has a convergent subsequence and the PS condition is satisfied. $\square$

It remains to show that we can obtain a path γ for $\mathcal{F}_\lambda$ under the critical level c^*. In order to get such a path we will take test functions of the form

$$\tilde{u}_\varepsilon = M\phi_\varepsilon,$$

where

$$\phi_\varepsilon = \varphi_{j,R}\, u_{j,\varepsilon}, \tag{4.2.11}$$

with $\varphi_{j,R}$ a cut-off function defined as (4.2.9) for some $R > 0$ small enough, $M > 0$ a large enough constant such that $\mathcal{F}_\lambda(\tilde{u}_\varepsilon) < 0$ and $u_{j,\varepsilon}$ are the family of functions

$$u_{j,\varepsilon}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - x_j|^2}\right)^{\frac{N-2}{2}}. \tag{4.2.12}$$

Then, under the previous considerations we define the set of paths

$$\Gamma_\varepsilon := \{\gamma \in \mathcal{C}([0,1], H_0^1(\Omega))\,;\, \gamma(0) = 0,\ \gamma(1) = \tilde{u}_\varepsilon\},$$

and we consider the minimax values

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{F}_\lambda(\gamma(t)).$$

The final issue we must solve now is the fact that the levels c_ε are always below c^* for ε small enough. Let us notice that the functions $u_{j,\varepsilon}$ are the extremal functions for the Sobolev's inequality in $\mathbb{R}^N$ where the constant S_N is achieved (see Talenti [**83**] for further details), i.e.

$$\int_{\mathbb{R}^N} |\nabla u_{j,\varepsilon}|^2 dx = S_N \left(\int_{\mathbb{R}^N} |u_{j,\varepsilon}|^{p+1} dx\right)^{2/2^*}.$$

For the sake of simplicity we will consider $x_j = 0$, so that we will denote $\varphi_{j,R} = \varphi$ under the construction (4.2.9) and $u_{j,\varepsilon} = u_\varepsilon$, as well as we will assume the normalization

$$\|u_\varepsilon\|_{L^{2^*}(\Omega)} = 1, \tag{4.2.13}$$

so that the Sobolev constant is given by

$$S_N = \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx.$$

Under this considerations, it can be proved the following.

LEMMA 4.2.6 ([**26**], Lemma 1.1). *Let ϕ be the function denoted by* (4.2.11) *around the point $x_j = 0$. Then,*

$$\int_{\mathbb{R}^N} \phi_\varepsilon^2 dx = \begin{cases} C\varepsilon + O(\varepsilon^2) & \text{if} \quad N = 3, \\ \frac{C\varepsilon^2}{2}|\log \varepsilon| + O(\varepsilon^2) & \text{if} \quad N = 4, \\ C\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if} \quad N \geq 5. \end{cases} \tag{4.2.14}$$

Moreover,

$$\|\nabla \phi_\varepsilon\|_2^2 = S_N + O(\varepsilon^{N-2}). \tag{4.2.15}$$

REMARK 4.2.2. *Using similar arguments one could also estimate $\|\phi_\varepsilon\|_{L^{2^*}(\Omega)} \sim C$ however, it is simpler if we normalize it as done in* (4.2.13).

To carry out the analysis of the levels c_ε it remains to obtain estimates dealing with the term $\int_\Omega \phi_\varepsilon (-\Delta)^{-1} \phi_\varepsilon dx$. To this end, we prove the following.

LEMMA 4.2.7. *Let ϕ_ε be the function denoted by* (4.2.11) *around the point $x_j = 0$. Then, there exists a constant $C > 0$ independent of ε such that*

$$\int_\Omega \phi_\varepsilon (-\Delta)^{-1} \phi_\varepsilon dx > C\varepsilon^4 \quad \text{if } N = 6, \tag{4.2.16}$$

$$\int_\Omega \phi_\varepsilon (-\Delta)^{-1} \phi_\varepsilon dx > C\varepsilon^\mu \quad \text{if } N \geq 7, \tag{4.2.17}$$

where $\frac{N}{2} + 1 > \mu > 1 + \frac{N}{N-4}$.

PROOF. Let $v_\varepsilon(x) = (-\Delta)^{-1}\phi_\varepsilon(x)$ and note that because of the definition of the cut-off function (4.2.9), we can choose $v_\varepsilon(x)$ such that

$$\begin{cases} (-\Delta) v_\varepsilon = \phi_\varepsilon & \text{in } B_{2R}(0), \\ v_\varepsilon = 0 & \text{in } \partial B_{2R}(0). \end{cases}$$

Moreover, since $\phi_\varepsilon > 0$ in $B_{2R}(0)$, thanks to the Maximum Principle, it follows that $v_\varepsilon > 0$ in $B_{2R}(0)$. Now, let us notice that for any $x \in B_R(0)$ we have $\phi_\varepsilon(x) = u_\varepsilon(x)$ as well as

$$\frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1 + \left(\frac{R}{\varepsilon}\right)^2\right)^{\frac{N-2}{2}}} \leq u_\varepsilon(x) \leq \varepsilon^{-\frac{N-2}{2}}.$$

Next, take $\rho < \frac{R}{2}$ and consider the function $\widetilde{v}(x) = \frac{2}{N}\left(1-\left(\frac{|x|}{2\rho}\right)^2\right)_+$, where $(\cdot)_+$ stands for the positive part. Then, $\widetilde{v}$ satisfies the problem

$$\begin{cases} (-\Delta)\widetilde{v} = \frac{1}{\rho^2} & \text{in } B_{2\rho}(0), \\ \widetilde{v} = 0 & \text{in } \partial B_{2\rho}(0). \end{cases}$$

To apply a comparison principle we choose $\rho = \varepsilon^{\alpha}$, with $\alpha > 0$, such that

$$(-\Delta)\widetilde{v} \le (-\Delta)v_\varepsilon \qquad \text{in} \quad B_{2\rho}(0).$$

Then, given $\varepsilon > 0$ arbitrarily small, we distinguish two cases depending upon $\alpha \ge 1$ or $\alpha < 1$. In the first case, since

$$u_\varepsilon(x)\Big|_{x\in B_{2\rho}(0)} \ge \frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1+\left(\frac{2\rho}{\varepsilon}\right)^2\right)^{\frac{N-2}{2}}} = \frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1+4\varepsilon^{2(\alpha-1)}\right)^{\frac{N-2}{2}}} \ge c_1\varepsilon^{-\frac{N-2}{2}},$$

for a positive constant $c_1 < 1$, we need to choose α such that,

$$\frac{1}{\varepsilon^{2\alpha}} \le c_1\varepsilon^{-\frac{N-2}{2}}.$$

We conclude $2\alpha \le \frac{N-2}{2}$. Therefore, we obtain the range $1 \le \alpha \le \frac{N-2}{4}$, which necessarily requires $N \ge 6$. In the second case, $\alpha < 1$, since

$$u_\varepsilon(x)\Big|_{x\in B_{2\rho}(0)} \ge \frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1+4\varepsilon^{-2(1-\alpha)}\right)^{\frac{N-2}{2}}} \ge c_2\varepsilon^{-\frac{N-2}{2}+(1-\alpha)(N-2)},$$

for a positive constant $c_2 < \frac{1}{4}$, we need to choose α such that

$$\frac{1}{\varepsilon^{2\alpha}} \le c_2\varepsilon^{-\frac{N-2}{2}+(1-\alpha)(N-2)}.$$

Then, we obtain the condition $\alpha \ge \frac{1}{2} + \frac{1}{N-4}$ that, together with $\alpha < 1$, implies $N > 6$. Finally, by construction,

$$0 = \widetilde{v}(x)\Big|_{x\in\partial B_{2\rho}(0)} < v_\varepsilon(x)\Big|_{x\in\partial B_{2\rho}(0)}$$

Because of the Maximum Principle, we conclude that $v_\varepsilon(x) > \widetilde{v}(x)$ for $x \in B_{2\rho}(0)$ thus,

$$\begin{aligned} \int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx &\ge \int_{B_R(0)} u_\varepsilon(x)v_\varepsilon(x)dx > \int_{B_{2\rho}(0)} u_\varepsilon(x)\widetilde{v}(x)dx \\ &\ge \int_{B_\rho(0)} u_\varepsilon(x)\widetilde{v}(x)dx = \frac{2}{N}\int_{B_\rho(0)} u_\varepsilon(x)\left(1-\left(\frac{|x|}{2\rho}\right)^2\right)dx \\ &\ge \frac{3}{2N}\int_{B_\rho(0)} u_\varepsilon(x)dx. \end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{B_\rho(0)} u_\varepsilon(x)dx &= \varepsilon^{-\frac{N-2}{2}} \int_{B_\rho(0)} \frac{1}{\left(1+\left(\frac{|x|}{\varepsilon}\right)^2\right)^{\frac{N-2}{2}}}dx = \varepsilon^{-\frac{N-2}{2}} \int_0^\rho \frac{r^{N-1}}{\left(1+\left(\frac{r}{\varepsilon}\right)^2\right)^{\frac{N-2}{2}}}dr\\ &= \varepsilon^{-\frac{N-2}{2}+N-1} \int_0^\rho \frac{(r/\varepsilon)^{N-1}}{\left(1+\left(\frac{r}{\varepsilon}\right)^2\right)^{\frac{N-2}{2}}}dr = \varepsilon^{\frac{N}{2}+1} \int_0^{\rho/\varepsilon} \frac{s^{N-1}}{(1+s^2)^{\frac{N-2}{2}}}ds\\ &\geq c\varepsilon^{\frac{N}{2}+1} \int_0^{\rho/\varepsilon} s^{N-1}ds = c\varepsilon^{\frac{N}{2}+1}\left(\frac{\rho}{\varepsilon}\right)^N,\end{aligned}$$

for a positive constant c. Then, since we have chosen $\rho=\varepsilon^\alpha$, we obtain

$$\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx > C\varepsilon^{\frac{N}{2}+1+N(\alpha-1)} \quad \text{for } \alpha\geq 1, N\geq 6, \tag{4.2.18}$$

and

$$\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx > C\varepsilon^{\frac{N}{2}+1-N(1-\alpha)} \quad \text{for } 1>\alpha>\frac{1}{2}+\frac{1}{N-4}, N\geq 7. \tag{4.2.19}$$

Now, we note that for the range $\alpha\geq 1$ the value $\alpha=1$ provides us with the optimum estimate in (4.2.18) and, thus, from here we obtain

$$\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx > C\varepsilon^{\frac{N}{2}+1} \quad \text{for } N\geq 6. \tag{4.2.20}$$

Moreover, since $\frac{N}{2}+1>\frac{N}{2}+1-N(1-\alpha)$ for $1>\alpha>\frac{1}{2}+\frac{1}{N-4}$, inequality (4.2.19) provides a stronger bound than the one provided by inequality (4.2.20) for any $N\geq 7$. Thus, inequality (4.2.20) is only useful for $N=6$, from where we conclude (4.2.16). Finally, setting $\mu=\frac{N}{2}+1-N(1-\alpha)$ in (4.2.19), it follows that $\frac{N}{2}+1>\mu>1+\frac{N}{N-4}$, and we conclude (4.2.17). □

Next we perform the analysis of the levels c_ε, proving that, in fact, the levels c_ε are always below the critical level c^* provided $\varepsilon>0$ is small enough.

LEMMA 4.2.8. *Assume $p=2^*-1$ and $N>6$. Then, there exists $\varepsilon>0$ small enough such that,*

$$\sup_{0\leq t\leq 1}\mathcal{F}_\lambda(t\tilde{u}_\varepsilon)<\frac{1}{N}S_N^{N/2}.$$

PROOF. Using (4.2.15) in Lemma 4.2.6 and assuming the normalization (4.2.13), we find

$$\begin{aligned}g(t):=\mathcal{F}_\lambda(t\tilde{u}_\varepsilon) &= \frac{t^2M^2}{2}\|\nabla\phi_\varepsilon\|^2_{L^2(\Omega)}-\frac{t^2M^2\lambda}{2}\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx-\frac{t^{2^*}M^{2^*}}{2^*}\\ &= \frac{M^2}{2}\left(S_N+O(\varepsilon^{N-2})-\lambda F(\varepsilon)\right)t^2-\frac{M^{2^*}}{2^*}t^{2^*},\end{aligned}$$

where $F(\varepsilon) = \int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx$. It is clear that $\lim_{t\to\infty} g(t) = -\infty$ as well as that $g(t) > 0$ for $t > 0$ small enough, therefore, the function $g(t)$ possesses a maximum value at the point,

$$t_\varepsilon := \left(\frac{M^2\left(S_N + O(\varepsilon^{N-2}) - \lambda F(\varepsilon)\right)}{M^{2^*}}\right)^{\frac{1}{2^*-2}}.$$

Moreover, at this point t_ε we have,

$$g(t_\varepsilon) = \frac{1}{N}\left(S_N + O(\varepsilon^{N-2}) - \lambda F(\varepsilon)\right)^{N/2}.$$

Then, the proof will be completed if the inequality

$$\frac{1}{N}\left(S_N + O(\varepsilon^{N-2}) - \lambda F(\varepsilon)\right)^{N/2} < \frac{1}{N}S_N^{N/2},$$

or, equivalently, the inequality

$$O(\varepsilon^{N-2}) < \lambda F(\varepsilon), \tag{4.2.21}$$

holds true provided ε is small enough. Moreover, because of (4.2.17) in Lemma 4.2.7, we have that $F(\varepsilon) > C\varepsilon^\mu$ with $\frac{N}{2} + 1 > \mu > 1 + \frac{N}{N-4}$. To finish the proof, let us show that, in fact, the stronger inequality

$$O(\varepsilon^{N-2}) < C\varepsilon^\mu, \tag{4.2.22}$$

holds true provided ε is small enough. To that end is enough to observe that (4.2.22) requires $N-2 > \mu$ that, together $\frac{N}{2}+1 > \mu > 1+\frac{N}{N-4}$, provides us with the condition $1+\frac{N}{N-4} < N-2$ which is equivalent to $(N-2)(N-6) > 0$, that is obviously satisfied. Thus, inequality (4.2.21) is satisfied provided ε is small enough. □

REMARK 4.2.3. *In the proof of Lemma 4.2.8 we proved that, for $N > 6$, $O(\varepsilon^{N-2}) < C\varepsilon^\mu$ provided ε is small enough and, because of* (4.2.17) *in Lemma 4.2.7, we concluded $O(\varepsilon^{N-2}) < C\varepsilon^\mu < F(\varepsilon)$. If we take $N = 6$ and we repeat the steps above, we readily find that* (4.2.16) *in Lemma 4.2.7 lead us to prove $O(\varepsilon^4) < C\varepsilon^4$, that can not be ensured either $\varepsilon > 0$ arbitrarily small or not. As we will see below (see Lemma 4.3.4), this restriction on the dimension is not a merely consequence of the accuracy of the estimates in Lemma 4.2.7.*

PROOF OF THEOREM 4.1.2. Thanks to Lemma 4.2.2 and Lemma 4.2.8, we find that

$$0 < c_\varepsilon \leq \sup_{0\leq t\leq 1} \mathcal{F}_\lambda(t\tilde{u}_\varepsilon) < \frac{1}{N}S_N^{N/2},$$

provided $\varepsilon > 0$ is small enough. Because of Lemma 4.2.2 the functional $\mathcal{F}_\lambda$ has the MP geometry. Moreover, because of Lemma 4.2.5 the functional $\mathcal{F}_\lambda$ satisfies the PS condition for any level c_ε provided $\varepsilon > 0$ is small enough. Therefore, we can apply the Mountain Pass Theorem to obtain the existence of a critical point $u \in H_0^1(\Omega)$. The rest follows as in the subcritical case. □

4.3. Existence of positive solutions for the system (S_λ)

In this section we prove the existence of positive solutions for the system (S_λ). We start by stating the analogous results of those obtained for the functional $\mathcal{F}_\lambda$.

LEMMA 4.3.1. *The functional $\mathcal{J}_\lambda$ denoted by* (4.1.5) *has the MP geometry.*

PROOF. Let us consider, without loss of generality, a pair $(g,h) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that $\|g\|_{L^{p+1}(\Omega)} = 1$. Then, taking a real number $t > 0$ and using the Young's inequality together with the Poincaré inequality and the Sobolev inequality (4.1.2), we find,

$$\begin{aligned}
\mathcal{J}_\lambda(tg,th) =& \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx + \frac{t^2}{2}\int_\Omega |\nabla h|^2 dx - t^2\sqrt{\lambda}\int_\Omega gh\, dx - \frac{t^{p+1}}{p+1} \\
\geq& \frac{t^2}{2}\left(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)} - \sqrt{\lambda}\int_\Omega g^2 dx - \sqrt{\lambda}\int_\Omega h^2 dx\right) - \frac{t^{p+1}}{p+1} \\
\geq& \frac{t^2}{2}\left(1 - \frac{\sqrt{\lambda}}{\lambda_1}\right)\left(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)}\right) - \|g\|^2_{H_0^1(\Omega)}\frac{C}{p+1}t^{p+1} \\
\geq& \left(\frac{1}{2}\left(1 - \frac{\sqrt{\lambda}}{\lambda_1}\right)t^2 - \frac{C}{p+1}t^{p+1}\right)\left(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)}\right),
\end{aligned} \tag{4.3.1}$$

where λ_1 is the first eigenvalue of the Laplace operator under Dirichlet boundary conditions. Since $0 < \lambda < \lambda_{1,2} = \lambda_1^2$ it follows that $\sqrt{\lambda} < \lambda_1$ and we obtain $\left(1 - \frac{\sqrt{\lambda}}{\lambda_1}\right) > 0$. Therefore, taking $t > 0$ such that,

$$0 < t^{p-1} < \frac{p+1}{2C}\left(1 - \frac{\sqrt{\lambda}}{\lambda_1}\right),$$

from (4.3.1) we conclude

$$\mathcal{J}_\lambda(tg,th) > 0.$$

Thus, the functional $\mathcal{J}_\lambda$ has a local minimum at $(u,v) = (0,0)$, i.e.,

$$\mathcal{J}_\lambda(tg,th) > \mathcal{J}_\lambda(0,0) = 0,$$

for any pair $(g,h) \in H_0^1(\Omega) \times H_0^1(\Omega)$ provided $t > 0$ is small enough. Also, it is clear that, because of the Poincaré inequality,

$$\begin{aligned}
\mathcal{J}_\lambda(tg,th) =& \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx + \frac{t^2}{2}\int_\Omega |\nabla h|^2 dx - t^2\sqrt{\lambda}\int_\Omega gh\, dx - \frac{t^{p+1}}{p+1} \\
\leq& \frac{t^2}{2}\left(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)} + \sqrt{\lambda}\int_\Omega g^2 dx + \sqrt{\lambda}\int_\Omega h^2 dx\right) - \frac{t^{p+1}}{p+1} \\
\leq& \frac{t^2}{2}\left(1 + \frac{\sqrt{\lambda}}{\lambda_1}\right)\left(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)}\right) - \frac{t^{p+1}}{p+1}.
\end{aligned}$$

Then,

$$\mathcal{J}_\lambda(tg,th) \to -\infty, \quad \text{as } t \to \infty,$$

and thus, there exists a pair $(\hat{u},\hat{v})$ such that $\mathcal{J}_\lambda(\hat{u},\hat{v}) < 0$. □

LEMMA 4.3.2. *Let $\{(u_n,v_n)\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ be a PS sequence at level c for the functional $\mathcal{J}_\lambda$, i.e.*

$$\mathcal{J}_\lambda(u_n,v_n) \to c, \quad \mathcal{J}'_\lambda(u_n,v_n) \to 0, \quad \textit{as} \quad n \to \infty.$$

Then,

$$\{(u_n,v_n)\} \quad \textit{is bounded in} \quad H_0^1(\Omega) \times H_0^1(\Omega).$$

PROOF. Since $\mathcal{J}'_\lambda(u_n,v_n)\to 0$ in $\big(H^1_0(\Omega)\times H^1_0(\Omega)\big)'$, in particular

$$\left\langle \mathcal{J}'_\lambda(u_n,v_n)\Big|\frac{(u_n,v_n)}{\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}}\right\rangle \to 0.$$

Thus, for any $\varepsilon>0$, there exists a subsequence, denoted again by $\{(u_n,v_n)\}$, such that,

$$\int_\Omega |\nabla u_n|^2dx+\int_\Omega |\nabla v_n|^2dx-2\sqrt{\lambda}\int_\Omega u_nv_ndx-\int_\Omega |u_n|^{p+1}dx=\Big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\Big]\cdot o(1).$$

Moreover, since $\mathcal{J}_\lambda(u_n,v_n)\to c$,

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2dx+\frac{1}{2}\int_\Omega |\nabla v_n|^2dx-\sqrt{\lambda}\int_\Omega u_nv_ndx-\frac{1}{p+1}\int_\Omega |u_n|^{p+1}dx=c+o(1),$$

for $n>0$ big enough. Therefore, for a positive constant μ (to be determined below) we find that

$$\mathcal{J}_\lambda(u_n,v_n)-\mu\left\langle \mathcal{J}'_\lambda(u_n,v_n)\Big|\frac{1}{\|u_n\|_{H^1_0(\Omega)}}(u_n,v_n)\right\rangle = c+\Big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\Big]\cdot o(1).$$

That is,

$$\left(\frac{1}{2}-\mu\right)\left[\int_\Omega |\nabla u_n|^2dx+\int_\Omega |\nabla v_n|^2dx\right]-(1-2\mu)\sqrt{\lambda}\int_\Omega u_nv_ndx-\left(\frac{1}{p+1}-\mu\right)\int_\Omega |u_n|^{p+1}dx$$
$$=c+\Big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\Big]\cdot o(1).$$

Hence, taking μ such that $\frac{1}{p+1}<\mu<\frac{1}{2}$,

$$\left(\frac{1}{2}-\mu\right)\left[\int_\Omega |\nabla u_n|^2dx+\int_\Omega |\nabla v_n|^2dx\right]-(1-2\mu)\sqrt{\lambda}\int_\Omega u_nv_ndx\le c+\Big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\Big]\cdot o(1),$$

and using Young's inequality,

$$\left(\frac{1}{2}-\mu\right)\left[\int_\Omega |\nabla u_n|^2dx+\int_\Omega |\nabla v_n|^2dx-\sqrt{\lambda}\int_\Omega u_n^2dx-\sqrt{\lambda}\int_\Omega v_n^2dx\right]$$
$$\le c+\Big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\Big]\cdot o(1).$$

Then, because of the Poincaré inequality, we conclude

$$\left(\frac{1}{2}-\mu\right)\left(1-\frac{\sqrt{\lambda}}{\lambda_1}\right)\Big[\|u_n\|^2_{H^1_0(\Omega)}+\|v_n\|^2_{H^1_0(\Omega)}\Big]\le c+\Big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\Big]\cdot o(1), \tag{4.3.2}$$

where λ_1 is the first eigenvalue of the Laplace operator under Dirichlet boundary conditions. Since $0<\lambda<\lambda_{1,2}=\lambda_1^2$, it follows that

$$\left(\frac{1}{2}-\mu\right)\left(1-\frac{\sqrt{\lambda}}{\lambda_1}\right)>0,$$

and thus, by (4.3.2), we conclude that the sequence $\{(u_n,v_n)\}$ is bounded in $H^1_0(\Omega)\times H^1_0(\Omega)$. $\square$

PROOF OF THEOREM 4.1.3. If $1<p<2^*-1$, given a PS sequence $\{(u_n,v_n)\}\subset H_0^1(\Omega)\times H_0^1(\Omega)$ at level c, by Lemma 4.3.1, the functional $\mathcal{J}_\lambda$ has the MP geometry. Moreover, by Lemma 4.3.2 and the compact inclusion

$$H_0^1(\Omega)\times H_0^1(\Omega)\hookrightarrow L^{p+1}(\Omega)\times L^{p+1}(\Omega),\quad \text{for } 2\le p+1<2^*,$$

provided by Rellich-Kondrachov Theorem, the functional $\mathcal{J}_\lambda$ satisfies the PS condition at any level c. Therefore, the hypotheses of the Mountain Pass Theorem are fulfilled and we conclude that the functional $\mathcal{J}_\lambda$ possesses a critical point $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$. Moreover, if we define the set of the paths

$$\Gamma:=\left\{\gamma\in\mathcal{C}\left([0,1],H_0^1(\Omega)\times H_0^1(\Omega)\right)\,;\,\gamma(0)=(0,0),\ \gamma(1)=(\hat{u},\hat{v})\right\},$$

with $(\hat{u},\hat{v})$ given as in the proof of Lemma 4.3.1, then

$$\mathcal{J}_\lambda(u,v)=c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}_\lambda(\gamma(t)).$$

To show the positivity of the pair (u,v) we argue as in the proof of Theorem 4.1.1. Let us consider the functional,

$$\mathcal{J}_\lambda^+(u,v)=\mathcal{J}_\lambda(u^+,v^+),$$

where, as before, $u^+=\max\{u,0\}$. Repeating with minor changes the arguments carried out above for the functional $\mathcal{J}_\lambda$ we conclude that the functional $\mathcal{J}_\lambda^+$ has a critical point $(\tilde{u},\tilde{v})$ such that $\tilde{u}\ge 0$ and $\tilde{v}\ge 0$. Moreover, by the Maximum Principle, it follows $\tilde{u}>0$ and $\tilde{v}>0$ and the proof is complete. □

4.3.1. Concentration-Compactness for the system (S_λ).

To prove the PS condition at the critical exponent case $p+1=2^*$ we must apply once again a concentration-compactness argument. This is done in several steps as performed above for the nonlocal problem (P_λ).

LEMMA 4.3.3. *Assume $p=2^*-1$. Then, the functional $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition for any level c such that,*

$$c<c^*=\frac{1}{N}S_N^{N/2}.$$

PROOF. Let $\{(u_n,v_n)\}\subset H_0^1(\Omega)\times H_0^1(\Omega)$ be a PS sequence of level $c<c^*$ for the functional $\mathcal{J}_\lambda$. Thanks to Lemma 4.3.2, the sequence $\{(u_n,v_n)\}$ is uniformly bounded and, as a consequence, we can assume that there exists a subsequence still denoted by $\{(u_n,v_n)\}$, such that,

$$\begin{aligned}
&(u_n,v_n)\rightharpoonup(u_0,v_0)\quad \text{weakly in}\quad H_0^1(\Omega)\times H_0^1(\Omega),\\
&(u_n,v_n)\to(u_0,v_0)\quad \text{strongly in}\quad L^q(\Omega)\times L^q(\Omega), 1\le q<2^*,\\
&(u_n,v_n)\to(u_0,v_0)\quad \text{a.e. in}\quad \Omega.
\end{aligned}\tag{4.3.3}$$

Moreover, we can assume that, up to a subsequence, there exist three measures μ, $\tilde{\mu}$ and ν such that $|\nabla u_n|^2$, $|\nabla v_n|^2$ and $|u_n|^{2^*}$, converge in the sense of the measures μ, $\tilde{\mu}$ and ν respectively. Thus, because of Lemma 4.2.4, there is a countable set I of points $\{x_j\}_{j\in I}\subset\overline{\Omega}$, and some positive numbers μ_j, $\tilde{\mu}_j$ and ν_j such that

$$
\begin{aligned}
&|\nabla u_n|^2 \rightharpoonup d\mu = |\nabla u_0|^2 + \sum_{j\in I}\mu_j\delta_{x_j},\\
&|\nabla v_n|^2 \rightharpoonup d\tilde{\mu} = |\nabla v_0|^2 + \sum_{j\in I}\tilde{\mu}_j\delta_{x_j},\\
&|u_n|^{2^*} \rightharpoonup d\nu = |u_0|^{2^*} + \sum_{j\in I}\nu_j\delta_{x_j},
\end{aligned}
\tag{4.3.4}
$$

where δ_{x_j} is the Dirac's delta centered at x_j with $j\in I$ and satisfying

$$
\mu_j \geq S_N \nu_j^{2/2^*}. \tag{4.3.5}
$$

Next, for $j\in I$, let $\varphi_{j,\varepsilon}\in C_0^\infty(\Omega)$ be a cut-off function satisfying (4.2.9) centered at $x_j\in\overline{\Omega}$. Thus, using $(\varphi_{j,\varepsilon}u_n, \varphi_{j,\varepsilon}v_n)$ as a test function, we find,

$$
\begin{aligned}
\langle \mathcal{J}_\lambda'(u_n,v_n)|(\varphi_{j,\varepsilon}u_n,\varphi_{j,\varepsilon}v_n)\rangle =& \int_\Omega \nabla u_n\cdot\nabla(\varphi_{j,\varepsilon}u_n)dx + \int_\Omega \nabla v_n\cdot\nabla(\varphi_{j,\varepsilon}v_n)dx - 2\sqrt{\lambda}\int_\Omega \varphi_{j,\varepsilon}u_nv_ndx\\
&- \int_\Omega \varphi_{j,\varepsilon}u_n^{2^*}dx\\
=& \int_\Omega \varphi_{j,\varepsilon}|\nabla u_n|^2dx + \int_\Omega \varphi_{j,\varepsilon}|\nabla v_n|^2dx - \int_\Omega \varphi_{j,\varepsilon}u_n^{2^*}dx\\
&+ \int_\Omega u_n\langle\nabla u_n,\nabla\varphi_{j,\varepsilon}\rangle dx + \int_\Omega v_n\langle\nabla v_n,\nabla\varphi_{j,\varepsilon}\rangle dx - 2\sqrt{\lambda}\int_\Omega \varphi_{j,\varepsilon}u_nv_ndx.
\end{aligned}
$$

Moreover, due to (4.3.3) and (4.3.4),

$$
\begin{aligned}
\lim_{n\to\infty}\langle \mathcal{J}_\lambda'(u_n,v_n)|(\varphi_{j,\varepsilon}u_n,\varphi_{j,\varepsilon}v_n)\rangle =& \int_\Omega \varphi_{j,\varepsilon}d\mu + \int_\Omega \varphi_{j,\varepsilon}d\tilde{\mu} - \int_\Omega \varphi_{j,\varepsilon}d\nu\\
&-2\sqrt{\lambda}\int_\Omega \varphi_{j,\varepsilon}u_0v_0dx + \int_\Omega u_0\left\langle\nabla u_0,\nabla\varphi_{j,\varepsilon}\right\rangle dx + \int_\Omega v_0\left\langle\nabla v_0,\nabla\varphi_{j,\varepsilon}\right\rangle dx.
\end{aligned}
$$

By construction,

$$
\lim_{\varepsilon\to0}\left[-2\sqrt{\lambda}\int_\Omega \varphi_{j,\varepsilon}u_0v_0dx + \int_\Omega u_0\left\langle\nabla u_0,\nabla\varphi_{j,\varepsilon}\right\rangle dx + \int_\Omega v_0\left\langle\nabla v_0,\nabla\varphi_{j,\varepsilon}\right\rangle dx\right] = 0.
$$

Then, as $\mathcal{J}_\lambda'(u_n)\to 0$ in $\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)'$, we obtain that,

$$
\lim_{\varepsilon\to0}\left(\int_\Omega \varphi_{j,\varepsilon}d\mu + \int_\Omega \varphi_{j,\varepsilon}d\tilde{\mu} - \int_\Omega \varphi_{j,\varepsilon}d\nu\right) = \mu_j + \tilde{\mu}_j - \nu_j = 0,
$$

and we conclude

$$
\nu_j = \mu_j + \tilde{\mu}_j. \tag{4.3.6}
$$

Finally, we have two options either the PS sequence has a convergent subsequence or it concentrates around some of the points x_j. In other words, $\nu_j = \mu_j = \tilde{\mu}_j = 0$, or there exists some $\nu_j > 0$ such that, by (4.3.5) and (4.3.6), $\nu_j \geq S_N^{N/2}$. In case of having concentration, we find that

$$\begin{aligned} c = \lim_{n\to\infty} \mathcal{J}_\lambda(u_n, v_n) &= \lim_{n\to\infty} \mathcal{J}_\lambda(u_n, v_n) - \frac{1}{2}\langle \mathcal{J}_\lambda(u_n, v_n) | (u_n, v_n)\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega |u_0|^{2^*} dx + \left(\frac{1}{2} - \frac{1}{2^*}\right) \nu_j \\ &\geq \frac{1}{N} S_N^{N/2} = c^*, \end{aligned}$$

in contradiction with the hypotheses $c < c^*$. Therefore, the PS sequence has a convergent subsequence and the PS condition is satisfied. □

The next step should be to show that we can obtain a path γ for $\mathcal{J}_\lambda$ under the critical level c^*. To obtain such a path we will assume test functions of the form

$$(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) = (M\phi_\varepsilon, M\rho\phi_\varepsilon),$$

where

$$\phi_\varepsilon = \varphi_{j,R}\, u_{j,\varepsilon},$$

with $\varphi_{j,R}$ is a cut-off function defined by (4.2.9), for some $R > 0$ small enough, $M > 0$ a sufficiently large constant such that $\mathcal{J}_\lambda(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) < 0$, ρ is a positive term to be determined below and $u_{j,\varepsilon}$ are the family of functions defined by (4.2.12). For the sake of simplicity, in the sequel we will consider $x_j = 0$ as well as the normalization (4.2.13).
Then, under the previous construction, we define the set of paths

$$\Gamma_\varepsilon := \left\{\gamma \in \mathcal{C}\left([0,1], H_0^1(\Omega) \times H_0^1(\Omega)\right) ; \gamma(0) = (0,0),\ \gamma(1) = (\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)\right\},$$

and consider the minimax value

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)).$$

Now we prove that, in fact, the levels c_ε are always below c^* for $\varepsilon > 0$ small enough.

Lemma 4.3.4. *Assume $p = 2^* - 1$. Then, there exists $\varepsilon > 0$ small enough such that,*

$$\sup_{0\leq t\leq 1} \mathcal{J}_\lambda(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) < \frac{1}{N} S_N^{N/2},$$

provided $N > 6$.

Proof. Let us denote by $F(\varepsilon)$ the estimate (4.2.14) in Lemma 4.2.6. Then, assuming the normalization (4.2.13),

$$\begin{aligned} g(t) := \mathcal{J}_\lambda(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) &= \left(\frac{t^2M^2}{2} + \frac{\rho^2t^2M^2}{2}\right) \|\nabla\phi_\varepsilon\|_{L^2(\Omega)}^2 - t^2M^2\rho\sqrt{\lambda}\int_\Omega \phi_\varepsilon^2 dx - \frac{t^{2^*}M^{2^*}}{2^*} \\ &= \frac{t^2M^2}{2}\left(\left(1+\rho^2\right)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\lambda}F(\varepsilon)\right) - \frac{t^{2^*}M^{2^*}}{2^*}. \end{aligned}$$

It is clear that $\lim\limits_{t\to\infty} g(t) = -\infty$, therefore, the function $g(t)$ possesses a maximum value at the point,

$$t_\varepsilon = \left(\frac{M^2\left[\left(1+\rho^2\right)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\lambda}F(\varepsilon)\right]}{M^{2^*}}\right)^{\frac{1}{2^*-2}}.$$

Moreover, at this point t_ε,

$$g(t_\varepsilon) = \frac{1}{N}\left[\left(1+\rho^2\right)\left[S_N + O(\varepsilon^{N-2})\right] - 2\rho\sqrt{\lambda}F(\varepsilon)\right]^{N/2}.$$

Then, the proof will be completed if we can choose $\rho > 0$ such that the inequality,

$$\left[\left(1+\rho^2\right)\left[S_N + O(\varepsilon^{N-2})\right] - 2\rho\sqrt{\lambda}F(\varepsilon)\right] < S_N, \tag{4.3.7}$$

holds true provided $\varepsilon > 0$ is small enough. Indeed, if we take $\rho = \varepsilon^\alpha$, with $\alpha > 0$ (to be determined), inequality (4.3.7) is equivalent to

$$S_N\varepsilon^{2\alpha} + O(\varepsilon^{N-2+2\alpha}) + O(\varepsilon^{N-2}) < 2\sqrt{\lambda}\varepsilon^\alpha F(\varepsilon),$$

Since $S_N\varepsilon^{2\alpha} + O(\varepsilon^{N-2+2\alpha}) + O(\varepsilon^{N-2}) = O(\varepsilon^\tau)$ with $\tau = \min\{2\alpha, N-2+2\alpha, N-2\} = \min\{2\alpha, N-2\}$, we are left to prove that we can choose $\alpha > 0$ such that,

$$O(\varepsilon^\tau) < 2\sqrt{\lambda}\varepsilon^\alpha \cdot \begin{cases} C\varepsilon + O(\varepsilon^2), & \text{if} \quad N=3, \\ \frac{C\varepsilon^2}{2}|\log\varepsilon| + O(\varepsilon^2), & \text{if} \quad N=4, \\ C\varepsilon^2 + O(\varepsilon^{N-2}), & \text{if} \quad N\geq 5. \end{cases} \tag{4.3.8}$$

provided $\varepsilon > 0$ is small enough.

- If $N = 3$, the corresponding inequality in (4.3.8) holds true if $\tau = \min\{2\alpha, 1\} > \alpha+1$ that is not possible.
- If $N = 4$, the corresponding inequality (4.3.8) holds true if
$$O(\varepsilon^\tau) < C\sqrt{\lambda}\varepsilon^{2\alpha+2}|\log\varepsilon| \quad \Rightarrow \quad O(\varepsilon^{\tau-2-\alpha}) < C\sqrt{\lambda}|\log\varepsilon|,$$
and thus, necessarily $\tau = \min\{2\alpha, 2\} > 2+\alpha$, that, once again, is not possible.
- If $N \geq 5$, the corresponding inequality (4.3.8) holds true if $\tau = \min\{2\alpha, N-2\} > 2+\alpha$. Let us observe that $\min\{a,b\} = \dfrac{1}{2}(a+b-|a-b|)$, hence, inequality (4.3.8) will be satisfied if we can choose $\alpha > 0$ such that
$$N - |2\alpha - (N-2)| > 6. \tag{4.3.9}$$
Now we have two options, either $2\alpha > N-2$ or $2\alpha < N-2$.
  - In the first case, thanks to inequality (4.3.9), we find the condition $\frac{N}{2}+1 > N-\alpha > 4$, that can be fulfilled only for $N > 6$.
  - In the second case, thanks to inequality (4.3.9), we find the condition $N-2 > 2\alpha > 4$, that can be fulfilled, once again, only for $N > 6$.

Thus, if $N > 6$ we can choose $\alpha > 2$ such that (4.3.8) is satisfied. Finally, note that with the assumption $\rho = \varepsilon^\alpha$ we have

$$t_\varepsilon = \left(\frac{M^2\left[\left(1+\rho^2\right)\left[S_N + O(\varepsilon^{N-2})\right] - 2\rho\sqrt{\lambda}F(\varepsilon)\right]}{M^{2^*}}\right)^{\frac{1}{2^*-2}} \geq \delta > 0,$$

provided $\varepsilon > 0$ is small enough. □

Proof of Theorem 4.1.4. Critical case. Thanks to Lemma 4.3.1 and Lemma 4.3.4,we find that

$$0 < c_\varepsilon \leq \sup_{0\leq t\leq 1} \mathcal{J}_\lambda(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) < \frac{1}{N}S_N^{N/2},$$

provided $\varepsilon > 0$ is small enough. Because of Lemma 4.3.1 the functional $\mathcal{J}_\lambda$ has the MPT geometry. Moreover, because of Lemma 4.3.3 the functional $\mathcal{J}_\lambda$ satisfies the PS condition for any level c_ε with $\varepsilon > 0$ small enough. Therefore, we can apply the Mountain Pass Theorem and conclude the existence of a critical point $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. The rest follows as in the subcritical case. □

4.4. Further Extensions

Let us consider the following high-order problem with generalized Navier boundary conditions,

$$(P_\lambda^{m+1}) \qquad \begin{cases} (-\Delta)^{m+1}u = \lambda u + (-\Delta)^m |u|^{p-1}u & \text{in} \quad \Omega \subset \mathbb{R}^N, \\ \quad (-\Delta)^j u = 0 \qquad \text{for } 0 \leq j \leq m, & \text{on} \quad \partial\Omega, \end{cases}$$

with a natural number m >1, and the variational problem obtained applying the operator $(-\Delta)^{-m}$ to (P_λ^{m+1}),

$$(E_\lambda^m) \qquad \begin{cases} -\Delta u = \lambda(-\Delta)^{-m}u + |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Associated with problem (E_λ^m) we consider the following Euler-Lagrange functional,

$$\mathcal{F}_{\lambda,m}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |(-\Delta)^{-m/2}u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

Note that, as it happens for $m = 1$, the embedding features for problem (E_λ^m) are governed by the standard second-order equation,

$$-\Delta u = |u|^{p-1}u,$$

thus, the variational framework coincides with the one of the case $m = 1$, so that we also consider $1 < p \leq 2^* - 1$.

Previously, to prove the existence of positive solutions to (P_λ^2), we first obtained the existence of a positive solution $u \in H_0^1(\Omega)$ to problem (P_λ) or, equivalently, for system (S_λ) and next, by standard elliptic regularity, we concluded u is also a positive solution to problem (P_λ^2). However, we can not repeat this scheme in the scenario of (P_λ^{m+1}) because, if we try to prove the existence of a positive solution to problem (E_λ^m) directly as performed for the problem (P_λ) in Section (4.2), we immediately run into complications: due to the lack of a comparison principle, we can not use a similar argument to Lemma (4.2.7) when dealing with the operator $(-\Delta)^{-m}$. Thus, we will make full use of the correspondence between problem (E_λ^m) and the following elliptic system,

$$(S_{\lambda,m}) \qquad \begin{cases} -\Delta u = \lambda^{\frac{1}{m+1}} v_1 + |u|^{p-1}u, \\ -\Delta v_1 = \lambda^{\frac{1}{m+1}} v_2, \\ -\Delta v_2 = \lambda^{\frac{1}{m+1}} v_3, \\ \quad \vdots \\ -\Delta v_m = \lambda^{\frac{1}{m+1}} u, \end{cases} \qquad \text{in} \quad \Omega, \quad (u, v_1, \ldots, v_m) = (0, 0, \ldots, 0) \quad \text{in } \partial\Omega,$$

whose associated Euler-Lagrange functional is defined by

$$\begin{aligned}\mathcal{J}_{\lambda,m}(\mathcal{U}) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\sum_{i=1}^{m}\int_\Omega |\nabla v_i|^2 dx \\ &\quad - \frac{\lambda^{\frac{1}{m+1}}}{m+1}\left(\int_\Omega u v_1 dx + \int_\Omega u v_m dx + \sum_{i=1}^{m-1}\int_\Omega v_i v_{i+1} dx\right) - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx,\end{aligned} \tag{4.4.1}$$

where $\mathcal{U} = (u, v_1, \ldots, v_m)$. The functional $\mathcal{J}_{\lambda,m}$ has the same structure as the functional $\mathcal{J}_\lambda$ thus, the ideas developed in Section 4.3 will fit, with slight variations, in this scenario. In particular, the main estimates needed in what follows have been already proven in Lemma 4.2.6.

Let us denote by $\lambda_{1,m+1}$ the first eigenvalue of the operator $(-\Delta)^{m+1}$ under the homogeneous generalized Navier boundary conditions given by (P_λ^{m+1}). It is clear from the spectral definition of the operator $(-\Delta)^{m+1}$ that $\lambda_{1,m+1} = \lambda_1^{m+1}$ with λ_1 the first eigenvalue of the Laplace operator under homogeneous Dirichlet boundary conditions.

The aim of this last section is then to prove the following.

THEOREM 4.4.1. *Assume $1 < p < 2^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,m+1})$, there exists at least a positive solution to system $(S_{\lambda,m})$.*

THEOREM 4.4.2. *Assume $p = 2^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,m+1})$, there exists at least a positive solution to system $(S_{\lambda,m})$ provided $N > 6$.*

We start determining the interval of values of the parameter $\lambda > 0$ compatible with existence of positive solutions related to problem (E_λ^m).

LEMMA 4.4.1. *Equation (E_λ^m) does not possess a positive solution when*

$$\lambda \geq \lambda_{1,m+1}.$$

PROOF. Using as a test function in (E_λ^m) the first eigenfunction φ_1 associated with the first eigenvalue λ_1 for the Laplacian operator $(-\Delta)$ with homogeneous Dirichlet boundary conditions together with $\lambda_{1,m+1} = \lambda_1^{m+1}$ the result follows. □

Next we deal with the MPT conditions. We state the analogous results to those of the case $m = 1$. Since the proofs of the next results rely on the ideas developed for the case $m = 1$, we will only remark the main differences, if any.

LEMMA 4.4.2. *The functional $\mathcal{J}_{\lambda,m}(\mathcal{U})$ has the MPT geometry.*

PROOF. The proof is similar to the proof of Lemma 4.3.1 so we omit the details. □

LEMMA 4.4.3. *Let $\mathbb{E}_m := H_0^1(\Omega) \times H_0^1(\Omega) \times \ldots \times H_0^1(\Omega)$ and assume that the sequence $\{\mathcal{U}_n\} = \{(u_n, v_{1,n}, \ldots, v_{m,n})\} \subset \mathbb{E}_m$ is a PS sequence for the functional $\mathcal{J}_{\lambda,m}$, i.e.*

$$\mathcal{J}_{\lambda,m}(\mathcal{U}_n) \to c, \quad \mathcal{J}'_{\lambda,m}(\mathcal{U}_n) \to 0, \quad \textit{as} \quad n \to \infty.$$

Then,

$$\{\mathcal{U}_n\} \quad \textit{is bounded in} \quad \mathbb{E}_m.$$

PROOF. Arguing as in the proof of Lemma 4.3.2 we find,

$$\begin{aligned}(m+1)\left(\frac{1}{2}-\mu\right)&\left(1-\frac{2\lambda^{\frac{1}{m+1}}}{(m+1)\lambda_1}\right)\left(\|u_n\|^2_{H_0^1(\Omega)}+\sum_{i=1}^m\|v_{i,n}\|^2_{H_0^1(\Omega)}\right)\\ &\leq (m+1)c+\left(\|u_n\|_{H_0^1(\Omega)}+\sum_{i=1}^m\|v_{i,n}\|_{H_0^1(\Omega)}\right)\cdot o(1).\end{aligned}$$

Keeping in mind Lemma 4.4.1, it follows that

$$\left(\frac{1}{2}-\mu\right)\left(1-\frac{2\lambda^{\frac{1}{m+1}}}{(m+1)\lambda_1}\right)>0,$$

and we conclude the boundedness of the sequence $\{\mathcal{U}_n\}$ in $\mathbb{E}_m$. $\square$

PROOF OF THEOREM 4.4.1. Combining Lemma 4.4.2 and Lemma 4.4.3 together with the Rellich-Kondrachov Theorem the hypotheses of the Mountain Pass Theorem are fulfilled and we conclude as in the proof of Theorem 4.1.3. $\square$

To finish, we deal with the critical case $p = 2^* - 1$. As it was done in previous sections, with the aid of a concentration-compactness argument we will prove that the PS condition is satisfied for any level below the critical level

$$c^* = \frac{1}{N}S_N^{N/2}.$$

Let us observe that the critical level c^* is independent of the order of the inverse operator involved in problem (E_λ^m) as it coincides with the critical level for problem (P_λ).

LEMMA 4.4.4. *The functional $\mathcal{J}_{\lambda,m}$ defined by* (4.4.1) *satisfies the Palais-Smale condition for any level c below the critical level c^*.*

PROOF. Let $\{\mathcal{U}_n\} = \{(u_n, v_{1,n}, \dots, v_{m,n})\} \subset \mathbb{E}_m$ be a PS sequence of level $c < c^*$. Because of Lemma 4.4.3 and Lemma 4.2.4, we can replicate the steps of the proof of Lemma 4.3.3 incorporating the slight difference that, instead (4.3.6), we find now

$$\nu_j = \mu_j + \sum_{i=1}^m \tilde{\mu}_{i,j}. \tag{4.4.2}$$

with

$$\mu_j \geq S_N \nu_j^{2/2^*}. \tag{4.4.3}$$

Then, either the PS sequence has a convergent subsequence or it concentrates around some of the points x_j. In other words, $\nu_j = \mu_j = \tilde{\mu}_{i,j} = 0$, or there exists some $\nu_j > 0$ such that, thanks to (4.4.2) and (4.4.3), $\nu_j \geq S_N^{N/2}$. In case of having concentration,

$$\begin{aligned}c = \lim_{n\to\infty}\mathcal{J}_{\lambda,m}(\mathcal{U}_n) &= \lim_{n\to\infty}\mathcal{J}_{\lambda,m}(\mathcal{U}_n) - \frac{1}{2}\langle \mathcal{J}_{\lambda,m}(\mathcal{U}_n)|\mathcal{U}_n\rangle\\ &= \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_\Omega |u_0|^{2^*}dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\nu_j\\ &\geq \frac{1}{N}S_N^{N/2} = c^*,\end{aligned}$$

in contradiction with the hypotheses $c < c^*$. $\square$

Finally, the only issue left to be proved is to show that we can obtain a path λ for the functional $\mathcal{J}_{\lambda,m}$ under the critical level c^*. Following the ideas of the previous sections, we will assume test functions of the form

$$\tilde{\mathcal{U}}_\varepsilon = (\tilde{u}_\varepsilon, \tilde{v}_{1,\varepsilon}, \ldots, \tilde{v}_{m,\varepsilon}) = (M\phi_\varepsilon, M\rho\phi_\varepsilon, \ldots, M\rho\phi_\varepsilon), \tag{4.4.4}$$

with $M > 0$ a sufficiently large constant so that $\mathcal{J}_{\lambda,m}(\tilde{\mathcal{U}}_\varepsilon) < 0$, ρ is positive term to be determined as in the case $m = 1$, and $u_{j,\varepsilon}$ are the family of functions defined by (4.2.12). As performed above we will consider $x_j = 0$. Then, under the previous construction, let us define the set of paths

$$\Gamma_\varepsilon := \{\gamma \in \mathcal{C}([0,1], \mathbb{E}_m)\,;\, \gamma(0) = \overline{0},\ \gamma(1) = \tilde{\mathcal{U}}_\varepsilon\},$$

and consider the minimax value

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{J}_{\lambda,m}(\gamma(t)).$$

Next, we check that any level c_ε is always below c^* provided $\varepsilon > 0$ is small enough. This is done thanks to Lemma 4.2.6.

Lemma 4.4.5. *Assume $p = 2^* - 1$ and $N > 6$. Then, there exists $\varepsilon > 0$ small enough such that,*

$$\sup_{0\le t\le 1} \mathcal{J}_{\lambda,m}(t\tilde{\mathcal{U}}_\varepsilon) < \frac{1}{N} S_N^{N/2}.$$

Proof. Let us denote by $F(\varepsilon)$ the estimate (4.2.14) in Lemma 4.2.6. Then, assuming the normalization (4.2.13), we get

$$\begin{aligned} g(t) :=& \mathcal{J}_{\lambda,m}(t\tilde{\mathcal{U}}_\varepsilon) \\ =& \left(\frac{1}{2}(1+m\rho^2)[S_N + O(\varepsilon^{N-2})] - \frac{\lambda^{\frac{1}{m+1}}}{m+1}(2\rho + (m-1)\rho^2)F(\varepsilon)\right) M^2t^2 - \frac{M^{2^*}t^{2^*}}{2^*}. \end{aligned}$$

Proceeding a in the proof of Lemma 4.3.4, we find that the proof will be completed if we can choose $\rho > 0$ such that the inequality,

$$O(\varepsilon^{N-2}) + m\rho^2 S_N + m\rho^2 O(\varepsilon^{N-2}) < 2\frac{\lambda^{\frac{1}{m+1}}}{m+1}(2\rho + (m-1)\rho^2)F(\varepsilon),$$

holds true provided $\varepsilon > 0$ is small enough. We take $\rho = \varepsilon^\alpha$ with $\alpha > 0$ (to be determined) and $\tau = \min\{N-2, 2\alpha, 2\alpha + N - 2\} = \min\{N-2, 2\alpha\}$. Then, since $O(\varepsilon^\alpha + \varepsilon^{2\alpha}) = O(\varepsilon^\alpha)$, we are left to prove that for a constant $C > 0$ the inequality,

$$O(\varepsilon^\tau) < C\varepsilon^\alpha F(\varepsilon), \tag{4.4.5}$$

holds true provided $\varepsilon > 0$ is small enough. Since inequality (4.4.5) coincides with (4.3.8) the arguments performed in Lemma 4.3.4 allow us to conclude. $\square$

Proof of Theorem 4.4.2. Thanks to Lemma 4.3.1 and Lemma 4.3.4,we find that

$$c_\varepsilon \le \sup_{t\ge 0} \mathcal{J}_\lambda(t\tilde{\mathcal{U}}_\varepsilon) < \frac{1}{N} S_N^{N/2},$$

provided $\varepsilon > 0$ is sufficiently small. Hence, combining Lemma 4.4.2 and Lemma 4.4.4 we can apply the Mountain Pass Theorem and conclude the existence of a critical point $\mathcal{U} \in \mathbb{E}_m$. The rest follows as in the former cases. $\square$

CHAPTER 5

Existence of positive solutions for a semilinear fractional elliptic equation involving an inverse fractional operator

In Chapter 5 we continue our study of elliptic problems involving inverse operators, extending the results of Chapter 4 to the fractional setting. As a natural generalization of problem (P_λ^2), we consider the fractional elliptic problem,

$$(P_\lambda^\alpha) \qquad \left\{ \begin{array}{ll} (-\Delta)^\alpha u = \lambda u + (-\Delta)^\beta |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ (-\Delta)^j u = 0, \text{ for } 0 \leq j < \lfloor \alpha \rfloor & \text{on } \partial\Omega, \end{array} \right.$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \beta < 1$, $\beta < \alpha < 1 + \beta$ and $\lambda > 0$.
Closely related to this problem, we study the existence of positive solutions for the fractional elliptic problem,

$$\left\{ \begin{array}{lll} (-\Delta)^{\alpha-\beta} u = \lambda(-\Delta)^{-\beta} u + |u|^{p-1}u & \text{in} & \Omega, \\ u = 0 & \text{on} & \partial\Omega, \end{array} \right.$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \beta < 1$, $\beta < \alpha < 1+\beta$ and $\lambda > 0$. We will prove that for the subcritical case $1 < p < 2_\mu^* - 1$, where $\mu := \alpha - \beta$ and $2_\mu^* = \frac{2N}{N-2\mu}$ is the critical exponent of the Sobolev embedding, there exists at least a positive solution if $\lambda < \lambda_{1,\alpha}$, denoting $\lambda_{1,\alpha}$ as the first eigenvalue of the fractional Laplace operator $(-\Delta)^\alpha$ under homogeneous boundary conditions. On the other hand, for the critical exponent case $p = 2_\mu^* - 1$, we will prove that there exists at least positive solution if $\lambda < \lambda_{1,\alpha}$ and $N > 4\alpha - 2\beta$. To obtain such a results we reformulate our problem in terms of a certain fractional elliptic cooperative system that allows us to overcome the difficulties coming from dealing with the inverse nonlocal term.

5.1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ with $N > 2\mu$ and

$$\mu := \alpha - \beta \quad \text{with} \quad 0 < \beta < 1 \quad \text{and} \quad \beta < \alpha < 1 + \beta.$$

We analyze the existence of positive solutions for the following fractional elliptic problem,

$$(P_\lambda^{\alpha,\beta}) \qquad \left\{ \begin{array}{lll} (-\Delta)^{\alpha-\beta} u = \lambda(-\Delta)^{-\beta} u + |u|^{p-1}u & \text{in} & \Omega, \\ u = 0 & \text{on} & \partial\Omega, \end{array} \right.$$

depending on the real parameter $\lambda > 0$. To this end, we consider,

$$1 < p \leq 2_\mu^* - 1 = \frac{N+2\mu}{N-2\mu},$$

where $2_\mu^* = \frac{2N}{N-2\mu}$ is the critical exponent of the Sobolev embedding. Associated with $(P_\lambda^{\alpha,\beta})$ we have the following Euler functional:

$$\mathcal{F}_\lambda(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{\mu}{2}} u|^2 dx - \frac{\lambda}{2}\int_\Omega |(-\Delta)^{-\frac{\beta}{2}} u|^2\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx, \tag{5.1.1}$$

such that the solutions of $(P_\lambda^{\alpha,\beta})$ can be obtained as critical points of the C^1 functional (5.1.1). Here, as customary, $(-\Delta)^{-\beta}u = w$, if

$$(-\Delta)^{\beta} w = u \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Note that $(-\Delta)^{-\beta}$ is a positive linear compact integral operator from $L^2(\Omega)$ into itself, well defined thanks to the Spectral Theorem. As performed in Chapter 1 for mixed boundary conditions, the definition of the fractional powers of the positive Laplace operator $(-\Delta)$, in a bounded domain Ω with homogeneous Dirichlet boundary data, is carried out through the spectral decomposition using the powers of the eigenvalues of $(-\Delta)$ with the same Dirichlet boundary condition (see for instance [**24**]). Thus, the fractional operator $(-\Delta)^{\mu}$, $0<\mu<1$, is well defined in the space of functions that vanish on the boundary,

$$H_0^{\mu}(\Omega) = \left\{ u = \sum_{j=1}^{\infty} a_j \varphi_j \in L^2(\Omega) : \ \|u\|_{H_0^{\mu}(\Omega)} = \left(\sum_{j=1}^{\infty} a_j^2 \lambda_j^{\mu} \right)^{\frac{1}{2}} < \infty \right\} = \overline{\mathcal{C}_0^{\infty}(\Omega)}^{\|\cdot\|_{H_0^{\mu}(\Omega)}}.$$

As a result of this definition it follows that,

$$\|u\|_{H_0^{\mu}(\Omega)} = \|(-\Delta)^{\frac{\mu}{2}} u\|_{L^2(\Omega)}, \tag{5.1.2}$$

as well as $(-\Delta)^{-\beta}u = \displaystyle\sum_{j=1}^{\infty} a_j \lambda_j^{-\beta}\varphi_j$. Next we introduce the definition of solution of $(P_\lambda^{\alpha,\beta})$.

DEFINITION 5.1.1. *We say that $u \in H_0^{\mu}(\Omega)$ is an energy or weak solution for problem $(P_\lambda^{\alpha,\beta})$ if,*

$$\int_{\Omega} (-\Delta)^{\frac{\mu}{2}} u (-\Delta)^{\frac{\mu}{2}} \phi dx = \lambda \int_{\Omega} (-\Delta)^{-\frac{\beta}{2}} u (-\Delta)^{-\frac{\beta}{2}} \phi dx + \int_{\Omega} |u|^{p-1} u \phi dx, \quad \forall \phi \in H_0^{\mu}(\Omega).$$

Equivalenty, $u \in H_0^{\mu}(\Omega)$ is a critical point of the functional defined by (5.1.1). We also observe that the functional embedding features for the equation in $(P_\lambda^{\alpha,\beta})$ are governed by the Sobolev's embedding Theorem. Let us recall the compact inclusion,

$$H_0^{\mu}(\Omega) \hookrightarrow L^{p+1}(\Omega), \quad 2 \le p+1 < 2_{\mu}^*, \tag{5.1.3}$$

being a continuous inclusion at the critical exponent $p = 2_{\mu}^* - 1$.
To define noninteger high-order powers for the Laplace operator, we follow the scheme of Chapter 4. Let us recall that the homogeneous Navier boundary conditions are defined as

$$u = \Delta u = \Delta^2 u = \ldots = \Delta^{k-1} u = 0, \quad \text{on } \partial\Omega.$$

The operator $(-\Delta)^{\alpha}$ is the α-th power of the classical Dirichlet Laplacian in the sense of the spectral theory and it can be defined as the operator whose action on a smooth function u is given by

$$\langle (-\Delta)^{\alpha} u, u \rangle = \sum_{j\ge 1} \lambda_j^{\alpha} |\langle u_1, \varphi_j \rangle|^2,$$

We refer to [**70, 71, 72**] for a complete study of this high-order fractional Laplace operator, referred as Navier fractional Laplacian, as well as useful properties of the fractional Sobolev space $H_0^{\alpha}(\Omega)$.

On the other hand, we have a connection between problem $(P_\lambda^{\alpha,\beta})$ and a fractional order elliptic system which turns out to be very useful in the sequel. In particular, taking $\omega := (-\Delta)^{-\beta}u$, problem $(P_\lambda^{\alpha,\beta})$ provides us with the fractional elliptic cooperative system,

$$\left\{\begin{array}{l}(-\Delta)^{\mu}u=\lambda\omega+|u|^{p-1}u,\\ (-\Delta)^{\beta}\omega=u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,\omega)=(0,0) \quad \text{in} \quad \partial\Omega. \tag{5.1.4}$$

Although, system (5.1.4) is not a variational system. In order to obtain a variational system from problem $(P_\lambda^{\alpha,\beta})$ we follow a similar idea to the one performed above, assuming whether $\alpha = 2\beta$ or $\alpha \neq 2\beta$. In the first case we split the parameter λ equally. Let us say, we take $v := \sqrt{\lambda}\omega$ and, recalling that $\mu := \alpha-\beta$, we obtain the following fractional elliptic cooperative system,

$$\left\{\begin{array}{l}(-\Delta)^{\beta}u=\sqrt{\lambda}v+|u|^{p-1}u,\\ (-\Delta)^{\beta}v=\sqrt{\lambda}u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{on} \quad \partial\Omega, \tag{S_λ^{β}}$$

whose associated energy functional is

$$\mathcal{J}_\lambda^{\beta}(u,v)=\frac{1}{2}\int_\Omega |(-\Delta)^{\frac{\beta}{2}}u|^2dx+\frac{1}{2}\int_\Omega |(-\Delta)^{\frac{\beta}{2}}v|^2dx-\sqrt{\lambda}\int_\Omega uvdx-\frac{1}{p+1}\int_\Omega |u|^{p+1}dx.$$

In the second case, $\alpha \neq 2\beta$, we split the parameter λ as follows,

$$\left\{\begin{array}{l}(-\Delta)^{\mu}u=\lambda^{1-\beta/\alpha}v+|u|^{p-1}u,\\ (-\Delta)^{\beta}v=\lambda^{\beta/\alpha}u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{on} \quad \partial\Omega. \tag{5.1.5}$$

Since the system (5.1.5) is still not variational, we transform it into the following variational system,

$$\left\{\begin{array}{l}\frac{1}{\lambda^{1-\beta/\alpha}}(-\Delta)^{\mu}u=v+\frac{1}{\lambda^{1-\beta/\alpha}}|u|^{p-1}u,\\ \frac{1}{\lambda^{\beta/\alpha}}(-\Delta)^{\beta}v=u,\end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v)=(0,0) \quad \text{on} \quad \partial\Omega. \tag{$S_\lambda^{\alpha,\beta}$}$$

whose associated functional is

$$\begin{aligned}\mathcal{J}_\lambda^{\alpha,\beta}(u,v)=&\frac{1}{2\lambda^{1-\beta/\alpha}}\int_\Omega |(-\Delta)^{\frac{\mu}{2}}u|^2dx+\frac{1}{2\lambda^{\beta/\alpha}}\int_\Omega |(-\Delta)^{\frac{\beta}{2}}v|^2dx-\int_\Omega uvdx\\ &-\frac{1}{(p+1)\lambda^{1-\beta/\alpha}}\int_\Omega |u|^{p+1}dx.\end{aligned}$$

Dealing with problem $(P_\lambda^{\alpha,\beta})$ presents some difficulties besides those that could naturally appear when we consider the critical exponent $p = 2_\mu^*-1$. Namely, to handle the inverse term $(-\Delta)^{-\beta}$ and the typical difficulties that arise when working with fractional nonlocal operators.

We will use the equivalence between problem $(P_\lambda^{\alpha,\beta})$ and systems (S_λ^{β}) and $(S_\lambda^{\alpha,\beta})$ to surpass the difficulties that arise while working with the inverse fractional Laplace operator $(-\Delta)^{-\beta}$. In particular, this approach will help us to avoid ascertaining explicit estimations for this inverse term. On the other hand, to overcome the usual difficulties that appear when dealing with fractional Laplace operators we will make full use of the extension technique giving an equivalent definition of the fractional operator $(-\Delta)^{\mu}$ in a bounded domain Ω by means of an auxiliary problem. Following the ideas developed in previous chapters, let us consider the cylinder $\mathscr{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}_+^{N+1}$ called extension cylinder. Moreover, let us denote by (x,y) the points belonging to $\mathscr{C}_\Omega$ and with $\partial_L\mathscr{C}_\Omega = \partial\Omega\times(0,\infty)$ the lateral boundary of the extension cylinder. Thus, given a function $u \in H_0^\mu(\Omega)$, define the μ-extension function w, denoted by $w := E_\mu[u]$, as the solution to problem,

$$\left\{\begin{array}{rll} -\operatorname{div}(y^{1-2\mu}\nabla w) = 0 & \text{in} & \mathscr{C}_\Omega, \\ w = 0 & \text{on} & \partial_L\mathscr{C}_\Omega, \\ w(x,0) = u(x) & \text{in} & \Omega \times \{y=0\}. \end{array}\right.$$

This extension function $w := E_\mu[u]$ belongs to the space

$$\mathcal{X}_0^\mu(\mathscr{C}_\Omega) = \overline{\mathcal{C}_0^\infty(\Omega \times [0,\infty))}^{\|\cdot\|_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}}, \text{ with } \|w\|^2_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)} = \kappa_\mu \int_{\mathscr{C}_\Omega} y^{1-2\mu}|\nabla w(x,y)|^2 dxdy.$$

According to [**24**], with that constant κ_μ, the extension operator is an isometry between $H_0^\mu(\Omega)$ and $\mathcal{X}_0^\mu(\mathscr{C}_\Omega)$, i.e.

$$\|E_\mu[\varphi]\|_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)} = \|\varphi\|_{H_0^\mu(\Omega)}, \text{ for all } \varphi \in H_0^\mu(\Omega). \tag{5.1.6}$$

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function through the formula

$$\frac{\partial w}{\partial \nu^\mu} := -\kappa_\mu \lim_{y\to 0^+} y^{1-2\mu}\frac{\partial w}{\partial y} = (-\Delta)^\mu u(x).$$

Thanks to the arguments shown above, we can reformulate problem $(P_\lambda^{\alpha,\beta})$ in terms of the extension problem as follows,

$$(\tilde{P}_\lambda) \qquad \left\{\begin{array}{rll} -\operatorname{div}(y^{1-2\mu}\nabla w) = 0 & \text{in} & \mathscr{C}_\Omega, \\ w = 0 & \text{on} & \partial_L\mathscr{C}_\Omega, \\ \frac{\partial w}{\partial \nu^\mu} = \lambda(-\Delta)^{-\beta}w + |w|^{p-1}w & \text{in} & \Omega \times \{y=0\}. \end{array}\right.$$

An energy solution of this problem is a function $w \in \mathcal{X}_0^\mu(\mathscr{C}_\Omega)$ such that

$$\kappa_\mu \int_{\mathscr{C}_\Omega} y^{1-2\mu}\langle \nabla w, \nabla\varphi\rangle dxdy = \int_\Omega \left(\lambda(-\Delta)^{-\beta}w + |w|^{p-1}w\right)\varphi(x,0)dx,$$

for any test function $\varphi \in \mathcal{X}_0^\mu(\mathscr{C}_\Omega)$. For any energy solution $w \in \mathcal{X}_0^\mu(\mathscr{C}_\Omega)$ to problem $(\tilde{P}_\lambda)$, the function $u = Tr[w] = w(\cdot,0)$ belongs to the space $H_0^\mu(\Omega)$ and it is an energy solution for the problem $(P_\lambda^{\alpha,\beta})$, and vice versa, if $u \in H_0^\mu(\Omega)$ is an energy solution to $(P_\lambda^{\alpha,\beta})$, then $w := E_\mu[u] \in \mathcal{X}_0^\mu(\mathscr{C}_\Omega)$ is an energy solution for $(\tilde{P}_\lambda)$ and, as a consequence, both formulations are equivalent. Finally, the energy functional associated with problem $(\tilde{P}_\lambda)$ is

$$\widetilde{\mathcal{F}}_\lambda(w) = \frac{\kappa_\mu}{2}\int_{\mathscr{C}_\Omega} y^{1-2\mu}|\nabla w|^2 dxdy - \frac{\lambda}{2}\int_\Omega |(-\Delta)^{-\frac{\beta}{2}}w|^2dx - \frac{1}{p+1}\int_\Omega |w|^{p+1}dx.$$

Since the extension function is an isometry, critical points of $\widetilde{\mathcal{F}}_\lambda$ in $\mathcal{X}_0^\mu(\mathscr{C}_\Omega)$ correspond to critical points of the functional $\mathcal{F}_\lambda$ in $H_0^\mu(\Omega)$. However, we can say even more. Indeed, arguing as in [**18**, Proposition 3.1], the minima of $\widetilde{\mathcal{F}}_\lambda$ also correspond to the minima of the functional $\mathcal{F}_\lambda$.

Another useful tool to be applied throughout this work will be the following trace inequality,

$$\int_{\mathscr{C}_\Omega} y^{1-2\mu}|\nabla z(x,y)|^2dxdy \geq C\left(\int_\Omega |z(x,0)|^r dx\right)^{\frac{2}{r}}, \tag{5.1.7}$$

for $1 \le r \le \frac{2N}{N-2\mu}$, $N > 2\mu$, and any $z \in \mathcal{X}_0^\mu(\mathscr{C}_\Omega)$. We observe then that inequality (5.1.7) is equivalent to the fractional Sobolev inequality,

$$\int_\Omega |(-\Delta)^{\mu/2} v|^2 dx \ge C \left(\int_\Omega |v|^r dx \right)^{\frac{2}{r}}, \tag{5.1.8}$$

for $1 \le r \le \frac{2N}{N-2\mu}$, $N > 2\mu$, and any $v \in H_0^\mu(\Omega)$.

Remark 5.1.1. *When $r = 2_\mu^*$, the best constant in (5.1.7) will be denoted by $S(\mu, N)$. This constant is independent of the domain Ω. Indeed, its exact value is given by the expression*

$$S(\mu, N) = \frac{2\pi^\mu \Gamma(1-\mu)\Gamma(\frac{N+2\mu}{2})(\Gamma(\frac{N}{2}))^{\frac{2\mu}{N}}}{\Gamma(\mu)\Gamma(\frac{N-2\mu}{2})(\Gamma(N))^\mu},$$

and it is never achieved when Ω is a bounded domain. Thus, we have,

$$\int_{\mathscr{C}_\Omega} y^{1-2\mu} |\nabla z(x,y)|^2 dxdy \ge S(\mu, N) \left(\int_\Omega |z(x,0)|^{\frac{2N}{N-2\mu}} dx \right)^{\frac{N-2\mu}{N}} \quad z \in \mathcal{X}_0^\mu(\mathbb{R}_+^{N+1}).$$

In the case when $\Omega = \mathbb{R}^N$, the constant $S(\mu, N)$ is achieved at $z = E_\mu[v]$ with

$$v(x) = v_{\mu,\varepsilon}(x) = \frac{\varepsilon^{\frac{N-2\mu}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2\mu}{2}}}, \tag{5.1.9}$$

with arbitrary $\varepsilon > 0$; see [**24**] *for further details. Finally, combining the previous comments the best constant in* (5.1.8) *with $\Omega = \mathbb{R}^N$ is given then by $\kappa_\mu S(\mu, N)$.*

Although systems (S_λ^β) and $(S_\lambda^{\alpha,\beta})$ no longer contain an inverse term as $(-\Delta)^{-\beta}$ they still are nonlocal systems, with all the complications that this entails. However, we use the extension technique shown above to reformulate the nonlocal systems (S_λ^β) and $(S_\lambda^{\alpha,\beta})$ in terms of the following local systems. Taking $w := E_\mu[u]$ and $z := E_\beta[v]$, the extension system corresponding to (S_λ^β) reads,

$$(\widetilde{S}_\lambda^\beta) \qquad \left\{ \begin{array}{lll} -\mathrm{div}(y^{1-2\beta}\nabla w) = 0 & \text{in} & \mathscr{C}_\Omega, \\ -\mathrm{div}(y^{1-2\beta}\nabla z) = 0 & \text{in} & \mathscr{C}_\Omega, \\ \dfrac{\partial w}{\partial \nu^\beta} = \sqrt{\lambda} z + |w|^{p-1} w & \text{in} & \Omega \times \{y=0\}, \\ \dfrac{\partial z}{\partial \nu^\beta} = \sqrt{\lambda} w & \text{in} & \Omega \times \{y=0\}, \\ w = z = 0 & \text{on} & \partial_L \mathscr{C}_\Omega, \end{array} \right.$$

whose associated functional is

$$\begin{aligned} \mathcal{H}_\lambda^\beta(w,z) = {} & \frac{\kappa_\beta}{2} \int_{\mathscr{C}_\Omega} y^{1-2\beta} |\nabla w|^2 dxdy + \frac{\kappa_\beta}{2} \int_{\mathscr{C}_\Omega} y^{1-2\beta} |\nabla z|^2 dxdy - \sqrt{\lambda} \int_\Omega w(x,0) z(x,0) dx \\ & - \frac{1}{p+1} \int_\Omega |w(x,0)|^{p+1} dx. \end{aligned}$$

Since the extension function is an isometry, critical points for the functional $\mathcal{H}_\lambda^\beta$ in $\mathcal{X}_0^\beta(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$ correspond to critical points of $\mathcal{J}_\lambda^\beta$ in $H_0^\beta(\Omega) \times H_0^\beta(\Omega)$. Moreover, arguing as in

[**18**, Proposition 3.1], the minima of $\mathcal{H}_\lambda^\beta$ also correspond to the minima of $\mathcal{J}_\lambda^\beta$. Similarly, the extension system of system $(S_\lambda^{\alpha,\beta})$ reads,

$$(\widetilde{S}_\lambda^{\alpha,\beta}) \qquad \left\{\begin{array}{lll} -\mathrm{div}(y^{1-2\mu}\nabla w)=0 & \text{in} & \mathscr{C}_\Omega, \\ -\mathrm{div}(y^{1-2\beta}\nabla z)=0 & \text{in} & \mathscr{C}_\Omega, \\ \dfrac{1}{\lambda^{1-\beta/\alpha}}\dfrac{\partial w}{\partial\nu^\mu}=z+\dfrac{1}{\lambda^{1-\beta/\alpha}}|w|^{p-1}w & \text{in} & \Omega\times\{y=0\}, \\ \dfrac{1}{\lambda^{\beta/\alpha}}\dfrac{\partial z}{\partial\nu^\beta}=w & \text{in} & \Omega\times\{y=0\}, \\ w=z=0 & \text{on} & \partial_L\mathscr{C}_\Omega, \end{array}\right.$$

whose associated functional is

$$\begin{aligned}\mathcal{H}_\lambda^{\alpha,\beta}(w,z)=&\frac{\kappa_\mu}{2\lambda^{1-\beta/\alpha}}\int_{\mathscr{C}_\Omega}y^{1-2\mu}|\nabla w|^2dxdy+\frac{\kappa_\beta}{2\lambda^{\beta/\alpha}}\int_{\mathscr{C}_\Omega}y^{1-2\beta}|\nabla z|^2dxdy\\ &-\int_\Omega w(x,0)z(x,0)dx-\frac{1}{(p+1)\lambda^{1-\beta/\alpha}}\int_\Omega w(x,0)^{p+1}dx.\end{aligned}$$

Once again, since the extension function is an isometry, critical points of $\mathcal{H}_\lambda^{\alpha,\beta}$ in $\mathcal{X}_0^\mu(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega)$ correspond to critical points of $\mathcal{J}_\lambda^{\alpha,\beta}$ in $H_0^\mu(\Omega)\times H_0^\beta(\Omega)$, and also, minima of $\mathcal{H}_\lambda^{\alpha,\beta}$ correspond to minima of $\mathcal{J}_\lambda^{\alpha,\beta}$.

Before finishing this introductory section, let us observe that problem $(P_\lambda^{\alpha,\beta})$ can be seen as a linear perturbation of the critical problem,

$$\left\{\begin{array}{lll} (-\Delta)^\mu u=|u|^{2_\mu^*-2}u & \text{in} & \Omega, \\ u=0 & \text{on} & \partial\Omega, \end{array}\right. \tag{5.1.10}$$

for which, after applying a Pohozaev-type result [**24**, Proposition 5.5], one can prove the non-existence of positive solutions under the star-shapeness assumption on the domain Ω. Moreover, the limit case $\beta\to 0$ in problem $(P_\lambda^{\alpha,\beta})$,

$$\left\{\begin{array}{rll} (-\Delta)^\alpha u= & \lambda u+|u|^{2_\alpha^*-2}u & \text{in } \Omega, \\ u= & 0 & \text{on } \partial\Omega, \end{array}\right. \qquad \text{with} \quad 0<\alpha<1, \tag{5.1.11}$$

is analyzed in [**18**], where the authors proved the existence of positive solutions for $N\geq 4\alpha$ if and only if $0<\lambda<\lambda_{1,\alpha}$, with $\lambda_{1,\alpha}$ being first eigenvalue of the $(-\Delta)^\alpha$ operator under homogeneous Dirichlet boundary conditions. Note that in our situation the nonlocal term $\lambda(-\Delta)^{-\beta}u=\lambda v$ plays actually the role of λu in [**18**].

Main results. We ascertain the existence of positive solutions for the problem $(P_\lambda^{\alpha,\beta})$ depending on the positive real parameter λ. To do so, we follow the sketch performed in Chapter 4: first we show the interval of the parameter λ for which there is the possibility of having positive solutions and next we use the equivalence between $(P_\lambda^{\alpha,\beta})$ and the systems (S_λ^β) and $(S_\lambda^{\alpha,\beta})$. Using the Mountain Pass Theorem [**12**], we will prove that there exists at least a positive solution for $(P_\lambda^{\alpha,\beta})$ with

$$0<\lambda<\lambda_{1,\alpha},$$

where $\lambda_{1,\alpha}$ is the first eigenvalue of the operator $(-\Delta)^\alpha$ under homogeneous Dirichlet boundary conditions. If $1<p+1<2_\mu^*$ one might apply the Mountain Pass Theorem directly since, as we will show, our problem possesses the mountain pass geometry and thanks to the compact embedding (5.1.3) the Palais-Smale condition is satisfied for the functionals $\mathcal{F}_\lambda$, $\mathcal{J}_\lambda^\beta$ and $\mathcal{J}_\lambda^{\alpha,\beta}$ (see details below in Section 5.2). However, at the critical exponent $p = 2_\mu^* - 1$, the compactness of the Sobolev embedding is lost and the problem becomes very delicate. To overcome this lack of compactness we apply a concentration-compactness argument relying on [**18**, Theorem 5.1], which is an adaptation to the fractional setting of the classical result of P.L. Lions, [**65**]. Then we are capable of proving that, under certain conditions, the Palais-Smale condition is satisfied for the functionals $\mathcal{H}_\lambda^\beta$ and $\mathcal{H}_\lambda^{\alpha,\beta}$. Thus, by the arguments above, the result will also follow for the functionals $\mathcal{F}_\lambda$, $\mathcal{J}_\lambda^\beta$ and $\mathcal{J}_\lambda^{\alpha,\beta}$. Consequently, we state now the main results of this chapter.

THEOREM 5.1.1. *Assume $1 < p < 2_\mu^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,\alpha})$, where $\lambda_{1,\alpha}$ is the first eigenvalue of $(-\Delta)^\alpha$ under homogeneous Dirichlet boundary conditions, there exists at least a positive solution for the problem $(P_\lambda^{\alpha,\beta})$.*

THEOREM 5.1.2. *Assume $p = 2_\mu^* - 1$. Then, for every $\lambda \in (0, \lambda_{1,\alpha})$, where $\lambda_{1,\alpha}$ is the first eigenvalue of $(-\Delta)^\alpha$ under homogeneous Dirichlet boundary conditions, there exists at least a positive solution for the problem $(P_\lambda^{\alpha,\beta})$ provided that $N > 4\alpha - 2\beta$.*

The phenomena observed in Chapter 4 about the effect of the nonlocal term on the existence issues becomes now clearer as shows Theorem 5.1.2, which addresses dimensions $N > 4\alpha - 2\beta$, in contrast to the existence result [**18**, Theorem 1.2] about the linear perturbation (5.1.11), that covers the wider range $N \geq 4\alpha$. In other words, the nonlocal term $(-\Delta)^{-\beta}u$, despite of being just a linear perturbation, has an important effect on the dimensions for which the classical technique based on the minimizers of the Sobolev constant still works. See details in Section 5.3, Lemma 5.3.6.

5.2. Subcritical exponent case

In this section we carry out the proof of Theorem 5.1.1. This is done through the equivalence between problem $(P_\lambda^{\alpha,\beta})$ and systems (S_λ^β) and $(S_\lambda^{\alpha,\beta})$. We note that the results proved in the sequel for the functionals $\mathcal{F}_\lambda$, $\mathcal{J}_\lambda^\beta$ and $\mathcal{J}_\lambda^{\alpha,\beta}$ translate immediately in analogous results for the functionals $\mathcal{H}_\lambda^\beta$ and $\mathcal{H}_\lambda^{\alpha,\beta}$. First, we characterize the existence of positive solutions for problem $(P_\lambda^{\alpha,\beta})$ in terms of the parameter λ. Moreover, for such characterization the following eigenvalue problem will be considered

$$(-\Delta)^\mu u = \lambda(-\Delta)^{-\beta}u. \tag{5.2.1}$$

Thus, we find that for the first eigenfunction ϕ_1 associated with the first eigenvalue λ_1^* of (5.2.1) under homogeneous Dirichlet boundary conditions

$$\int_\Omega |(-\Delta)^{\frac{\mu}{2}}\phi_1|^2 dx = \lambda_1^* \int_\Omega |(-\Delta)^{-\frac{\beta}{2}}\phi_1|^2 dx,$$

and therefore,

$$\lambda_1^* = \inf_{u \in H_0^\mu(\Omega)} \frac{\int_\Omega |(-\Delta)^{\frac{\mu}{2}}u|^2 dx}{\int_\Omega |(-\Delta)^{-\frac{\beta}{2}}u|^2 dx}. \tag{5.2.2}$$

On the other hand, thanks to the definition of the fractional operator $(-\Delta)^\mu$, we have that $\phi_1 \equiv \varphi_1$ with φ_1 being the first eigenfunction of the Laplace operator under homogeneous

Dirichlet boundary conditions. Then,

$$(-\Delta)^{\mu}\phi_1 = (-\Delta)^{\mu}\varphi_1 = \lambda_1^{\mu}\varphi_1 \quad \text{and} \quad (-\Delta)^{-\beta}\phi_1 = (-\Delta)^{-\beta}\varphi_1 = \lambda_1^{-\beta}\varphi_1,$$

with λ_1 being the first eigenvalue of the Laplace operator under homogeneous Dirichlet boundary conditions. Hence, due to (5.2.1), we conclude that $\lambda_1^* = \lambda_1^{\mu+\beta} = \lambda_1^{\alpha} = \lambda_{1,\alpha}$ and, thus, λ_1^* coincides with the first eigenvalue of the operator $(-\Delta)^{\alpha}$ under homogeneous Dirichlet boundary conditions. Subsequently, we have the following.

LEMMA 5.2.1. *Problem* $(P_{\lambda}^{\alpha,\beta})$ *does not possess a positive solution when*

$$\lambda \geq \lambda_{1,\alpha}.$$

PROOF. Assume that u is a positive solution of $(P_{\lambda}^{\alpha,\beta})$ and let φ_1 be a positive first eigenfunction of the Laplace operator in Ω under homogeneous Dirichlet boundary conditions. Taking φ_1 as a test function for equation $(P_{\lambda}^{\alpha,\beta})$ we obtain

$$\begin{aligned}\lambda_1^{\mu}\int_{\Omega} u\varphi_1 dx = \int_{\Omega}\varphi_1(-\Delta)^{\mu}u dx &= \lambda\int_{\Omega}\varphi_1(-\Delta)^{-\beta}u dx + \int_{\Omega}|u|^{p-1}u\varphi_1 dx \\ &> \lambda\int_{\Omega}\varphi_1(-\Delta)^{-\beta}u dx = \lambda\int_{\Omega} u(-\Delta)^{-\beta}\varphi_1 dx \\ &= \frac{\lambda}{\lambda_1^{\beta}}\int_{\Omega} u\varphi_1 dx.\end{aligned}$$

Hence, $\lambda_1^{\mu} > \frac{\lambda}{\lambda_1^{\beta}}$, and we conclude that $\lambda < \lambda_1^{\mu+\beta} = \lambda_1^{\alpha} = \lambda_{1,\alpha}$, proving the lemma. □

Because of Lemma 5.2.1 we shall assume from now on that $0 < \lambda < \lambda_{1,\alpha}$. Moreover, since our discussion is mainly based on the use of the Mountain Pass Theorem [**12**] we have to check that the functional $\mathcal{F}_{\lambda}$ has the appropriate geometry and also satisfies the Palais-Smale condition. As long as we can, we will prove these conditions for both the functional $\mathcal{F}_{\lambda}$ and the remaining functionals, otherwise we will use the one that suits better our situation.

LEMMA 5.2.2. *The functionals* $\mathcal{F}_{\lambda}$, $\mathcal{J}_{\lambda}^{\beta}$ *and* $\mathcal{J}_{\lambda}^{\alpha,\beta}$ *have the mountain pass geometry .*

PROOF. We start with the functional $\mathcal{F}_{\lambda}$. Without loss of generality, we consider a function $g \in H_0^{\mu}(\Omega)$ such that $\|g\|_{p+1} = 1$. Then, taking a real number $t > 0$ and using (5.2.2), the fractional Sobolev inequality (5.1.8) and (5.1.2), we find that,

$$\begin{aligned}\mathcal{F}_{\lambda}(tg) &= \frac{t^2}{2}\int_{\Omega}|(-\Delta)^{\frac{\mu}{2}}g|^2 dx - \frac{\lambda t^2}{2}\int_{\Omega}|(-\Delta)^{-\frac{\beta}{2}}g|^2 dx - \frac{t^{p+1}}{p+1} \\ &\geq \frac{t^2}{2}\int_{\Omega}|(-\Delta)^{\frac{\mu}{2}}g|^2 dx - \frac{\lambda t^2}{2\lambda_{1,\alpha}}\int_{\Omega}|(-\Delta)^{\frac{\mu}{2}}g|^2 dx - \frac{t^{p+1}}{p+1} \\ &\geq \frac{t^2}{2}\left(1 - \frac{\lambda}{\lambda_{1,\alpha}}\right)\int_{\Omega}|(-\Delta)^{\frac{\mu}{2}}g|^2 dx - \frac{t^{p+1}}{C(p+1)}\int_{\Omega}|(-\Delta)^{\frac{\mu}{2}}g|^2 dx \\ &= \|g\|_{H_0^{\mu}(\Omega)}^2\left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{1,\alpha}}\right)t^2 - \frac{1}{C(p+1)}t^{p+1}\right) > 0,\end{aligned}$$

with C a positive cosntant coming from inequality (5.1.7) and assuming t small enough, i.e., for $t>0$ such that,

$$0<t^{p-1}<\frac{C(p+1)}{2}\left(1-\frac{\lambda}{\lambda_{1,\alpha}}\right).$$

Furthermore, it is clear that

$$\begin{aligned}\mathcal{F}_\lambda(tg)&=\frac{t^2}{2}\int_\Omega|(-\Delta)^{\frac{\mu}{2}}g|^2dx-\frac{\lambda t^2}{2}\int_\Omega|(-\Delta)^{-\frac{\beta}{2}}g|^2dx-\frac{t^{p+1}}{p+1}\\&\leq\frac{t^2}{2}\|g\|^2_{H_0^\mu(\Omega)}-\frac{t^{p+1}}{p+1}.\end{aligned}$$

Then,

$$\mathcal{F}_\lambda(tg)\to-\infty,\quad\text{as}\quad t\to\infty,$$

and thus, there exists $\hat{u}\in H_0^\mu(\Omega)$ such that $\mathcal{F}_\lambda(\hat{u})<0$. Hence, the functional $\mathcal{F}_\lambda$ has the mountain pass geometry.

We continue with $\mathcal{J}_\lambda^\beta$, so that $\alpha=2\beta$. Let us consider, without loss of generality, a pair $(g,h)\in H_0^\beta(\Omega)\times H_0^\beta(\Omega)$ such that $\|g\|_{p+1}=1$. Then, taking $t>0$ small enough and using Young's inequality, the fractional Sobolev inequality (5.1.8) and (5.1.2), we find that,

$$\begin{aligned}\mathcal{J}_\lambda^\beta(tg,th)&=\frac{t^2}{2}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}g|^2dx+\frac{t^2}{2}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}h|^2dx-\sqrt{\lambda}t^2\int_\Omega ghdx-\frac{t^{p+1}}{p+1}\\&\geq\frac{t^2}{2}\left(\|g\|^2_{H_0^\beta(\Omega)}+\|h\|^2_{H_0^\beta(\Omega)}-\sqrt{\lambda}\int_\Omega g^2dx-\sqrt{\lambda}\int_\Omega h^2dx\right)-\frac{t^{p+1}}{p+1}\\&\geq\frac{t^2}{2}\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)\left(\|g\|^2_{H_0^\beta(\Omega)}+\|h\|^2_{H_0^\beta(\Omega)}\right)-\frac{t^{p+1}}{C(p+1)}\|g\|^2_{H_0^\beta(\Omega)}\\&\geq\left[\|g\|^2_{H_0^\beta(\Omega)}+\|h\|^2_{H_0^\beta(\Omega)}\right]\left(\frac{1}{2}\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)t^2-\frac{1}{C(p+1)}t^{p+1}\right)>0.\end{aligned}$$

The last inequality follows immediately since $\lambda<\lambda_{1,\alpha}=\lambda_1^\alpha=\lambda_1^{2\beta}$ thus, $\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)=\left(1-\frac{\sqrt{\lambda}}{\lambda_1^\beta}\right)>0$, and we can take $t>0$ small enough such that,

$$0<t^{p-1}<\frac{C(p+1)}{2}\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right).$$

Moreover, away from the trivial solution it follows that

$$\begin{aligned}\mathcal{J}_\lambda^\beta(tg,th)&=\frac{t^2}{2}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}g|^2dx+\frac{t^2}{2}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}h|^2dx-\sqrt{\lambda}t^2\int_\Omega ghdx-\frac{t^{p+1}}{p+1}\\&\leq\frac{t^2}{2}\|g\|^2_{H_0^\beta(\Omega)}+\frac{t^2}{2}\|h\|^2_{H_0^\beta(\Omega)}+\frac{\sqrt{\lambda}t^2}{2}\int_\Omega g^2dx+\frac{\sqrt{\lambda}t^2}{2}\int_\Omega h^2dx-\frac{t^{p+1}}{p+1}\\&\leq\frac{t^2}{2}\left(1+\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)\left[\|g\|^2_{H_0^\beta(\Omega)}+\|h\|^2_{H_0^\beta(\Omega)}\right]-\frac{t^{p+1}}{p+1}.\end{aligned}$$

Then,

$$\mathcal{J}_\lambda^\beta(tg,th)\to-\infty,\quad \text{as}\quad t\to\infty,$$

and there exists $(\hat{u},\hat{v})\in H_0^\beta(\Omega)\times H_0^\beta(\Omega)$ such that $\mathcal{J}_\lambda^\beta(\hat{u},\hat{v})<0$. Therefore, the functional $\mathcal{J}_\lambda^\beta$ has the mountain pass geometry.

Next, we deal with the functional $\mathcal{J}_\lambda^{\alpha,\beta}$. We assume that $(g,h)\in H_0^\mu(\Omega)\times H_0^\beta(\Omega)$ and $\|g\|_{p+1}=1$. Then, taking $t>0$ small enough and using Young's inequality, the fractional Sobolev inequality (5.1.8) and (5.1.2) it yields

$$\begin{aligned}
\mathcal{J}_\lambda^{\alpha,\beta}(tg,th)=&\frac{t^2}{2\lambda^{1-\beta/\alpha}}\int_\Omega|(-\Delta)^{\frac{\mu}{2}}g|^2dx+\frac{t^2}{2\lambda^{\beta/\alpha}}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}h|^2dx-t^2\int_\Omega gh\,dx\\
&-\frac{t^{p+1}}{(p+1)\lambda^{1-\beta/\alpha}}\\
\geq&\frac{t^2}{2}\left(\frac{1}{\lambda^{1-\beta/\alpha}}\|g\|^2_{H_0^\mu(\Omega)}+\frac{1}{\lambda^{\beta/\alpha}}\|h\|^2_{H_0^\beta(\Omega)}-\int_\Omega g^2dx-\int_\Omega h^2dx\right)\\
&-\frac{t^{p+1}}{(p+1)\lambda^{1-\beta/\alpha}}\\
\geq&\|g\|^2_{H_0^\mu(\Omega)}\left(\frac12\left(\frac{1}{\lambda^{1-\beta/\alpha}}-\frac{1}{\lambda_{1,\mu}}\right)t^2-\frac{1}{C(p+1)\lambda^{1-\beta/\alpha}}t^{p+1}\right)\\
&+\|h\|^2_{H_0^\beta(\Omega)}\left(\frac12\left(\frac{1}{\lambda^{\beta/\alpha}}-\frac{1}{\lambda_{1,\beta}}\right)t^2-\frac{1}{C(p+1)\lambda^{1-\beta/\alpha}}t^{p+1}\right)>0,
\end{aligned}$$

which is true since $\lambda<\lambda_{1,\alpha}=\lambda_1^\alpha$ and therefore $\lambda^{1-\beta/\alpha}<\lambda_1^{\alpha(1-\beta/\alpha)}=\lambda_1^\mu=\lambda_{1,\mu}$ as well as $\lambda^{\beta/\alpha}<\lambda_1^{\alpha(\beta/\alpha)}=\lambda_1^\beta=\lambda_{1,\beta}$. Hence we conclude, $\left(\frac{1}{\lambda^{1-\beta/\alpha}}-\frac{1}{\lambda_{1,\mu}}\right)>0$ and $\left(\frac{1}{\lambda^{\beta/\alpha}}-\frac{1}{\lambda_{1,\beta}}\right)>0$ and we can take $t>0$ small enough such that,

$$0<t^{p-1}<\frac{C(p+1)\lambda^{1-\beta/\alpha}}{2}\min\left\{\frac{1}{\lambda^{1-\beta/\alpha}}-\frac{1}{\lambda_{1,\mu}},\frac{1}{\lambda^{\beta/\alpha}}-\frac{1}{\lambda_{1,\beta}}\right\}.$$

Finally, arguing as in the previous cases,

$$\begin{aligned}
\mathcal{J}_\lambda^{\alpha,\beta}(tg,th)=&\frac{t^2}{2\lambda^{1-\beta/\alpha}}\int_\Omega|(-\Delta)^{\frac{\mu}{2}}g|^2dx+\frac{t^2}{2\lambda^{\beta/\alpha}}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}h|^2dx-t^2\int_\Omega gh\,dx\\
&-\frac{t^{p+1}}{(p+1)\lambda^{1-\beta/\alpha}}\\
\leq&\frac{t^2}{2}\left[\left(\frac{1}{\lambda^{1-\beta/\alpha}}+\frac{1}{\lambda_{1,\mu}}\right)\|g\|^2_{H_0^\beta(\Omega)}+\left(\frac{1}{\lambda^{\beta/\alpha}}+\frac{1}{\lambda_{1,\beta}}\right)\|h\|^2_{H_0^\beta(\Omega)}\right]\\
&-\frac{t^{p+1}}{(p+1)\lambda^{1-\beta/\alpha}}\\
\leq&\frac{t^2}{2}K(\lambda,\mu,\beta)\left[\|g\|^2_{H_0^\beta(\Omega)}+\|h\|^2_{H_0^\beta(\Omega)}\right]-\frac{t^{p+1}}{p+1}\|g\|^{p+1}_{L^{p+1}(\Omega)},
\end{aligned}$$

for $K(\lambda,\mu,\beta)=\max\left\{\frac{1}{\lambda^{1-\beta/\alpha}}+\frac{1}{\lambda_{1,\mu}},\frac{1}{\lambda^{\beta/\alpha}}+\frac{1}{\lambda_{1,\beta}}\right\}$. Then,

$$\mathcal{J}_\lambda^\beta(tg,th)\to-\infty,\quad \text{as}\quad t\to\infty,$$

so that, there exists $(\hat{u},\hat{v})\in H_0^\mu(\Omega)\times H_0^\beta(\Omega)$ such that $\mathcal{J}_\lambda^{\alpha,\beta}(\hat{u},\hat{v})<0$, and the proof of the lemma is completed. $\square$

LEMMA 5.2.3. *The functionals $\mathcal{H}_\lambda^\beta$ and $\mathcal{H}_\lambda^{\alpha,\beta}$ have the mountain pass geometry .*

PROOF. The proof is similar to the proof of Lemma 5.2.2, we only need to note that, because of [**24**, Lemma 2.4], the extension function minimizes the $\|\cdot\|_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}$ norm along all the functions with the same trace on $\{y=0\}$, i.e.,

$$\|E_\mu[\varphi(\cdot,0)]\|_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}\leq\|\varphi\|_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}\quad\text{for all}\quad\varphi\in\mathcal{X}_0^\mu(\mathscr{C}_\Omega).$$

Therefore,

$$\lambda_1^\mu=\inf_{\substack{u\in H_0^\mu(\Omega)\\ u\not\equiv0}}\frac{\|u\|^2_{H_0^\mu(\Omega)}}{\|u\|^2_{L^2(\Omega)}}=\inf_{\substack{w\in\mathcal{X}_0^\mu(\mathscr{C}_\Omega)\\ w\not\equiv0}}\frac{\|w\|^2_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^2(\Omega)}}. \tag{5.2.3}$$

Repeating the arguments shown in the proof of Lemma 5.2.2 the result follows easily. $\square$

Now we turn our attention to the so-called Palais-Smale condition.

DEFINITION 5.2.1. *Let V be a Banach space and $\{u_n\}\subset V$ a Palais-Smale (PS) sequence for a functional $\mathfrak{F}$, i.e.*

$$\mathfrak{F}(u_n)\quad\textit{is bounded and}\quad\mathfrak{F}'(u_n)\to0\quad\textit{in } V'\quad\textit{as}\quad n\to\infty, \tag{5.2.4}$$

where V' is the dual space of E. Then $\{u_n\}$ satisfies a PS condition if

$$\{u_n\}\quad\textit{has a convergent subsequence.} \tag{5.2.5}$$

In particular, given a PS sequence $\{u_n\}\subset V$ such that $\mathfrak{F}(u_n)\to c$, if (5.2.5) is satisfied, we will say that the PS sequence satisfies a PS condition at level c for the functional $\mathfrak{F}$. Moreover, we say that the functional $\mathfrak{F}$ satisfies the PS condition at level c if every PS sequence at level c for $\mathfrak{F}$ possesses a convergent subsequence in V.
In the subcritical range the PS condition is always satisfied at any level c due mostly to the compact embedding (5.1.3). However, at the critical exponent 2_μ^* the compactness in the Sobolev embedding is lost and, as a consequence, the PS condition will be satisfied only for levels c below certain critical level.

LEMMA 5.2.4. *Let $\{u_n\}$ be a PS sequence at level c for the functional $\mathcal{F}_\lambda$, i.e.*

$$\mathcal{F}_\lambda(u_n)\to c,\quad\mathcal{F}_\lambda'(u_n)\to0,\quad\textit{as}\quad n\to\infty.$$

Then, $\{u_n\}$ is bounded in $H_0^\mu(\Omega)$.

PROOF. Since $\mathcal{F}_\lambda'(u_n)\to0$ in $(H_0^\mu(\Omega))'$, in particular we have $\langle\mathcal{F}_\lambda'(u_n)|\frac{u_n}{\|u_n\|_{H_0^\mu(\Omega)}}\rangle\to0$. Thus, for any $\varepsilon>0$ there exists a subsequence, denoted again by $\{u_n\}$, such that for n big enough,

$$\int_\Omega|(-\Delta)^{\frac{\mu}{2}}u_n|^2dx-\lambda\int_\Omega|(-\Delta)^{-\frac{\beta}{2}}u_n|^2dx-\int_\Omega|u_n|^{p+1}dx=o(1)\cdot\|u_n\|_{H_0^\mu(\Omega)}.$$

Moreover, since $\mathcal{F}_\lambda(u_n)\to c$,

$$\frac12\int_\Omega|(-\Delta)^{\frac{\mu}{2}}u_n|^2dx-\frac{\lambda}{2}\int_\Omega|(-\Delta)^{-\frac{\beta}{2}}u_n|^2dx-\frac{1}{p+1}\int_\Omega|u_n|^{p+1}dx=c+o(1).$$

Thus, for a positive constant η (to be determined below) we find that
$$\mathcal{F}_\lambda(u_n)-\eta\langle\mathcal{F}_\lambda'(u_n)|u_n\rangle=c+o(1)\cdot\|u_n\|_{H_0^\mu(\Omega)}.$$
That is,
$$\left(\frac{1}{2}-\eta\right)\int_\Omega|(-\Delta)^{\frac{\mu}{2}}u_n|^2dx-\left(\frac{1}{2}-\eta\right)\int_\Omega|(-\Delta)^{-\frac{\beta}{2}}u_n|^2dx-\left(\frac{1}{p+1}-\eta\right)\int_\Omega|u_n|^{p+1}dx$$
$$=c+o(1)\cdot\|u_n\|_{H_0^\mu(\Omega)}.$$
Taking first η such that $\frac{1}{p+1}<\eta$ it follows that
$$\left(\frac{1}{2}-\eta\right)\int_\Omega|(-\Delta)^{\frac{\mu}{2}}u_n|^2dx-\left(\frac{1}{2}-\eta\right)\int_\Omega|(-\Delta)^{-\frac{\beta}{2}}u_n|^2dx\le c+o(1)\cdot\|u_n\|_{H_0^\mu(\Omega)}.$$
Therefore, by (5.2.2),
$$\left(\frac{1}{2}-\eta\right)\left(1-\frac{\lambda}{\lambda_{1,\alpha}}\right)\int_\Omega|(-\Delta)^{\frac{\mu}{2}}u_n|^2dx\le c+o(1)\cdot\|u_n\|_{H_0^\mu(\Omega)}.$$
Choosing as well $\eta<\frac{1}{2}$ and since $\lambda<\lambda_{1,\alpha}$, using (5.1.2) we conclude that
$$0<\left(\frac{1}{2}-\eta\right)\left(1-\frac{\lambda}{\lambda_{1,\alpha}}\right)\|u_n\|^2_{H_0^\mu(\Omega)}\le c+o(1)\cdot\|u_n\|_{H_0^\mu(\Omega)}.$$
Thus, the sequence $\{u_n\}$ is bounded in $H_0^\mu(\Omega)$. □

Now we turn our attention to the funcionals $\mathcal{J}_\lambda^\beta$ and $\mathcal{J}_\lambda^{\alpha,\beta}$.

LEMMA 5.2.5. *Let $\{(u_n,v_n)\}$ be a PS sequence at level c for the functional $\mathcal{J}_\lambda^\beta$, i.e.*
$$\mathcal{J}_\lambda^\beta(u_n,v_n)\to c,\quad \left(\mathcal{J}_\lambda^\beta\right)'(u_n,v_n)\to 0,\quad \text{as}\quad n\to\infty.$$
Then, $\{(u_n,v_n)\}$ is bounded in $H_0^\beta(\Omega)\times H_0^\beta(\Omega)$.

PROOF. Since $\left(\mathcal{J}_\lambda^\beta\right)'(u_n,v_n)\to 0$ in $\left(H_0^\beta(\Omega)\times H_0^\beta(\Omega)\right)'$ it follows that, in particular,
$$\left\langle\left(\mathcal{J}_\lambda^\beta\right)'(u_n,v_n)\Big|\frac{(u_n,v_n)}{\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}}\right\rangle\to 0.$$
Thus, for any $\varepsilon>0$ there exists a subsequence, denoted again by $\{(u_n,v_n)\}$, such that for n big enough,
$$\int_\Omega|(-\Delta)^{\frac{\beta}{2}}u_n|^2dx+\int_\Omega|(-\Delta)^{\frac{\beta}{2}}v_n|^2dx$$
$$-2\sqrt{\lambda}\int_\Omega u_nv_ndx-\int_\Omega|u_n|^{p+1}dx=o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).$$
Moreover, since $\mathcal{J}_\lambda^\beta(u_n,v_n)\to c$, we find that
$$\frac{1}{2}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}u_n|^2dx+\frac{1}{2}\int_\Omega|(-\Delta)^{\frac{\beta}{2}}v_n|^2dx-\sqrt{\lambda}\int_\Omega u_nv_ndx-\frac{1}{p+1}\int_\Omega|u_n|^{p+1}dx$$
$$=c+o(1).$$

Thus, for a positive constant η (to be determined below), we find that

$$\mathcal{J}_\lambda^\beta(u_n,v_n)-\eta\left\langle\left(\mathcal{J}_\lambda^\beta\right)'(u_n,v_n)|(u_n,v_n)\right\rangle=c+o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).$$

That is,

$$\begin{aligned}\left(\frac{1}{2}-\eta\right)\int_\Omega|(-\Delta)^{\frac{\beta}{2}}u_n|^2dx+\left(\frac{1}{2}-\eta\right)\int_\Omega|(-\Delta)^{\frac{\beta}{2}}v_n|^2dx&\\ -\sqrt{\lambda}(1-2\eta)\int_\Omega u_nv_ndx-\left(\frac{1}{p+1}-\eta\right)\int_\Omega|u_n|^{p+1}dx&\\ =c+o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).&\end{aligned}$$

Taking η such that $\frac{1}{p+1}<\eta$ it follows that

$$\begin{aligned}\left(\frac{1}{2}-\eta\right)\left(\int_\Omega|(-\Delta)^{\frac{\beta}{2}}u_n|^2dx+\int_\Omega|(-\Delta)^{\frac{\beta}{2}}v_n|^2dx\right)-\sqrt{\lambda}(1-2\eta)\int_\Omega u_nv_ndx&\\ \leq c+o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).&\end{aligned}$$

Moreover, using Young's inequality, we find,

$$\begin{aligned}\left(\frac{1}{2}-\eta\right)\left(\int_\Omega|(-\Delta)^{\frac{\beta}{2}}u_n|^2dx+\int_\Omega|(-\Delta)^{\frac{\beta}{2}}v_n|^2dx\right)-\frac{\sqrt{\lambda}(1-2\eta)}{2}\left(\int_\Omega u_n^2dx+\int_\Omega v_n^2dx\right)&\\ \leq c+o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right),&\end{aligned}$$

and therefore,

$$\begin{aligned}\left(\frac{1}{2}-\eta\right)\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)\left(\int_\Omega|(-\Delta)^{\frac{\beta}{2}}u_n|^2dx+\int_\Omega|(-\Delta)^{\frac{\beta}{2}}v_n|^2dx\right)&\\ \leq c+o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).&\end{aligned}$$

Moreover, taking $\eta<\frac{1}{2}$ and since $\lambda<\lambda_{1,\alpha}=\lambda_1^\alpha=\lambda_1^{2\beta}$, thanks to (5.1.2) we can conclude that

$$0<\left(\frac{1}{2}-\eta\right)\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)\left(\|u_n\|^2_{H_0^\beta(\Omega)}+\|v_n\|^2_{H_0^\beta(\Omega)}\right)\leq c+o(1)\left(\|u_n\|_{H_0^\beta(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).$$

Thus, the sequence $\{(u_n,v_n)\}$ is bounded in $H_0^\beta(\Omega)\times H_0^\beta(\Omega)$. □

LEMMA 5.2.6. *Let $\{(u_n,v_n)\}$ be a PS sequence at level c for the functional $\mathcal{J}_\lambda^{\alpha,\beta}$, i.e.*

$$\mathcal{J}_\lambda^{\alpha,\beta}(u_n,v_n)\to c,\quad \left(\mathcal{J}_\lambda^{\alpha,\beta}\right)'(u_n,v_n)\to 0,\quad as\quad n\to\infty.$$

Then, $\{(u_n,v_n)\}$ is bounded in $H_0^\mu(\Omega)\times H_0^\beta(\Omega)$.

Proof. Proceeding as in the proof of Lemma 5.2.5, we find that, for $\frac{1}{p+1}<\eta<\frac{1}{2}$,

$$\left(\frac{1}{2}-\eta\right)\left(\frac{1}{\lambda^{1-\beta/\alpha}}-\frac{1}{\lambda_{1,\mu}}\right)\|u_n\|^2_{H_0^\mu(\Omega)}+\left(\frac{1}{2}-\eta\right)\left(\frac{1}{\lambda^{\beta/\alpha}}-\frac{1}{\lambda_{1,\beta}}\right)\|v_n\|^2_{H_0^\beta(\Omega)}$$
$$\leq c+o(1)\left(\|u_n\|_{H_0^\mu(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).$$

Since $\lambda<\lambda_{1,\alpha}=\lambda_1^\alpha$, we get $\lambda^{1-\beta/\alpha}<\lambda_1^{\alpha-\beta}=\lambda_1^\mu=\lambda_{1,\mu}$ as well as $\lambda^{\beta/\alpha}<\lambda_1^\beta=\lambda_{1,\beta}$ and we can conclude

$$0<\left(\frac{1}{2}-\eta\right)\left(\frac{1}{\lambda^{1-\beta/\alpha}}-\frac{1}{\lambda_{1,\mu}}\right)\|u_n\|^2_{H_0^\mu(\Omega)}+\left(\frac{1}{2}-\eta\right)\left(\frac{1}{\lambda^{\beta/\alpha}}-\frac{1}{\lambda_{1,\beta}}\right)\|v_n\|^2_{H_0^\beta(\Omega)}$$
$$\leq c+o(1)\left(\|u_n\|_{H_0^\mu(\Omega)}+\|v_n\|_{H_0^\beta(\Omega)}\right).$$

□

At last we deal with the extension functionals $\mathcal{H}_\lambda^\beta$ and $\mathcal{H}_\lambda^{\alpha,\beta}$.

Lemma 5.2.7. *Let $\{(w_n,z_n)\}$ be a PS sequence at level c for the functional $\mathcal{H}_\lambda^\beta$ (resp. for the functional $\mathcal{H}_\lambda^{\alpha,\beta}$). Then, $\{(w_n,z_n)\}$ is bounded in $\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega)$ (resp. in $\mathcal{X}_0^\mu(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega)$)*

Proof. Taking in mind (5.2.3) and arguing as in the proof of Lemma 5.2.5 (resp. Lemma 5.2.6) we find

$$0<\left(\frac{1}{2}-\eta\right)\left(1-\frac{\sqrt{\lambda}}{\lambda_{1,\beta}}\right)\left(\|w_n\|^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)}+\|z_n\|^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)}\right)$$
$$\leq c+o(1)\left(\|w_n\|_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)}+\|z_n\|_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)}\right).$$

and

$$0<\left(\frac{1}{2}-\eta\right)\left(\frac{1}{\lambda^{1-\beta/\alpha}}-\frac{1}{\lambda_{1,\mu}}\right)\|w_n\|^2_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}+\left(\frac{1}{2}-\eta\right)\left(\frac{1}{\lambda^{\beta/\alpha}}-\frac{1}{\lambda_{1,\beta}}\right)\|z_n\|^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)}$$
$$\leq c+o(1)\left(\|w_n\|_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)}+\|z_n\|_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)}\right).$$

respectively. Therefore, the PS sequence is bounded in the corresponding space. □

Now, we are able to prove one of the main results of this chapter.

Proof of Theorem 5.1.1. Let us consider the subcritical case $1<p<2_\mu^*-1$. Given a PS sequence $\{u_n\}\in H_0^\mu(\Omega)$ at level c for the functional $\mathcal{F}_\lambda$, thanks to Lemma 5.2.4 and the compact inclusion (5.1.3), the PS condition is satisfied. Hence, the functional $\mathcal{F}_\lambda$ satisfies the PS condition at any level c. Moreover, by Lemma 5.2.2, the functional $\mathcal{F}_\lambda$ has the mountain pass geometry. Then, due to the Mountain Pass Theorem [**12**] the functional $\mathcal{F}_\lambda$ possesses a critical point $u\in H_0^\mu(\Omega)$. Moreover, if we define all the paths between the origin and $\hat{u}$ as

$$\Gamma:=\{\gamma\in C([0,1],H_0^\mu(\Omega))\,;\,\gamma(0)=0,\ \gamma(1)=\hat{u}\},$$

with $\hat{u}$ given as in Lemma 5.2.2, i.e. $\mathcal{F}_\lambda(\hat{u})<0$, then,

$$c:=\mathcal{F}_\lambda(u)=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{F}_\lambda(\gamma(t)).$$

Moreover, to show that $u>0$, let us consider the functional,

$$\mathcal{F}_\lambda^+(u)=\mathcal{F}_\lambda(u^+),$$

where $u^+=\max\{u,0\}$. Repeating with minor changes the arguments carried out above one readily shows that what was proved for the functional $\mathcal{F}_\lambda$ still holds for the functional $\mathcal{F}_\lambda^+$. Hence, it follows that $u\geq 0$ and by the Maximum Principle, $u>0$. Then, the proof of existence of positive solutions to $(P_\lambda^{\alpha,\beta})$ in the subcritical range $1<p<2_\mu^*-1$ is complete. Now we proof Theorem 5.1.1 by means of the systems (S_λ^β) and $(S_\lambda^{\alpha,\beta})$. Let us recall that this cases refer to $\alpha=2\beta$ and $\alpha\neq 2\beta$ respectively, so that $\mu=\beta$ and $\mu=\alpha-\beta$ respectively. Given a PS sequence $\{(u_n,v_n)\}\in H_0^\beta(\Omega)\times H_0^\beta(\Omega)$ at level c for the functional $\mathcal{J}_\lambda^\beta$, thanks to Lemma 5.2.5 and the compact inclusion

$$H_0^\beta(\Omega)\times H_0^\beta(\Omega)\hookrightarrow L^{p+1}(\Omega)\times L^{p+1}(\Omega),$$

provided by (5.1.3), we find that the PS condition is satisfied at any level c. Hence, the functional $\mathcal{J}_\lambda^\beta$ satisfies the PS condition. Moreover, by Lemma 5.2.2, the functional $\mathcal{J}_\lambda^\beta$ has the mountain pass geometry. Then, by the Mountain Pass Theorem the functional $\mathcal{J}_\lambda^\beta$ possesses a critical point $(u,v)\in H_0^\beta(\Omega)\times H_0^\beta(\Omega)$. Then, if we define the family of paths

$$\Gamma:=\{\gamma\in C([0,1],H_0^\beta(\Omega)\times H_0^\beta(\Omega))\,;\,\gamma(0)=(0,0),\ \gamma(1)=(\hat{u},\hat{v})\},$$

with $(\hat{u},\hat{v})$ such that $\mathcal{J}_\lambda^\beta(\hat{u},\hat{v})<0$, then,

$$c:=\mathcal{J}_\lambda^\beta(u,v)=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}_\lambda^\beta(\gamma(t)).$$

Again, to show that $(u,v)>(0,0)$, with $u>0$ and $v>0$, we might consider the functional,

$$\mathcal{J}_\lambda^{\beta,+}(u,v)=\mathcal{J}_\lambda^\beta(u^+,v^+),$$

where $u^+=\max\{u,0\}$ and $v^+=\max\{v,0\}$ and repeat the arguments shown above proving the positivity of the solutions. Consequently, the existence of solution for the system (S_λ^β) in the subcritical range $1<p<2_\beta^*-1$ is achieved.

Similarly, for the functional $\mathcal{J}_\lambda^{\alpha,\beta}$ we use that due to (5.1.3) the inclusion

$$H_0^\mu(\Omega)\times H_0^\beta(\Omega)\hookrightarrow L^{p+1}(\Omega)\times L^{q+1}(\Omega),$$

is compact for $q<2_\beta^*-1$, so that the PS condition is satisfied at any level c. The rest follows as in the previous cases. □

Remark 5.2.1. *The proof of Theorem 5.1.1 is mainly based on the compact inclusion* (5.1.3). *More specifically, thanks to* (5.1.3)*, the inclusion*

$$H_0^\mu(\Omega)\times H_0^\beta(\Omega)\hookrightarrow L^{p+1}(\Omega)\times L^{q+1}(\Omega), \tag{5.2.6}$$

is compact for $q<2_\beta^*-1$. In the case $\alpha=2\beta$ we have that $\mu=\beta$ and $q=p$ in (5.2.6)*. Then, repeating the scheme of the proof of Theorem 5.1.1 with the appropriate modifications, one can prove the existence of a positive solution for a system of the form*

$$\left\{\begin{array}{l}(-\Delta)^\beta u=\lambda v+|u|^{p-1}u,\\(-\Delta)^\beta v=\lambda u+|v|^{p-1}v,\end{array}\right.\quad \textit{in}\quad \Omega,\quad (u,v)=(0,0)\quad \textit{on}\quad \partial\Omega,$$

for a suitable $\lambda>0$. In case that $\alpha\neq 2\beta$ we have two options:

- *If $\alpha < 2\beta$ the situation is similar since, in this case $2_\mu^* := 2_{\alpha-\beta}^* < 2_\beta^*$. Then, taking $p < 2_\mu^* - 1 < 2_\beta^* - 1$, thanks to* (5.1.3) *the inclusion* (5.2.6) *is compact for $q < 2_\beta^* - 1$ and, in particular, for $q = p$. Then, following the argument performed above for Theorem* (5.1.1)*, one can prove the existence of a positive solution for a system of the form*

$$\left\{\begin{array}{l} (-\Delta)^{\alpha-\beta} u = \lambda_1 v + |u|^{p-1} u, \\ (-\Delta)^{\beta} v = \lambda_2 u + |v|^{p-1} v, \end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v) = (0,0) \quad \text{on} \quad \partial\Omega,$$

for a suitable pair (λ_1, λ_2).

- *On the other hand, if $\alpha > 2\beta$ then $2_{\alpha-\beta}^* > 2_\beta^*$ then, we have two more options*

$$\text{either} \quad p < 2_\beta^* - 1 < 2_{\alpha-\beta}^* - 1 \quad \text{or} \quad 2_\beta^* - 1 < p < 2_{\alpha-\beta}^* - 1.$$

In the first case, we also obtain that the inclusion (5.2.6) *is compact for $q = p$ whereas if $2_\beta^* < p + 1 < 2_{\alpha-\beta}^*$ the inclusion* (5.2.6) *is compact for $q < 2_\beta^* - 1 < p$. This range of values provides good enough functional embedding features to study systems like*

$$\left\{\begin{array}{l} (-\Delta)^{\alpha-\beta} u = \lambda_1 v + u^p, \\ (-\Delta)^{\beta} v = \lambda_2 u + v^q, \end{array}\right. \quad \text{in} \quad \Omega, \quad (u,v) = (0,0) \quad \text{on} \quad \partial\Omega,$$

by means of a similar scheme to that of Theorem 5.1.1. Finally, in the particular case $p = q$ the compact inclusion (5.2.6) *holds for $p < \min\{2_{\alpha-\beta}^*, 2_\beta^*\} - 1$.*

5.3. Concentration-Compactness at the critical exponent

In this section we focus on the critical exponent case, $p = 2_\mu^* - 1$, proving Theorem 5.1.2. Our aim is to prove the PS condition for the functional $\mathcal{F}_\lambda$ since the rest of the proof will be similar to what we performed in the previous section for the sub-critical case.
Due to the lack of compactness of the Sobolev embedding the study of the PS condition becomes a delicate task. We then follow the standard approach, namely, by means of a concentration-compactness argument we will obtain that the PS condition is satisfied at levels c below certain critical level c^* (to be determined) and later we construct a sequence whose energy is below that critical level c^*. Both steps are strongly based on the use of concrete test functions and on how the different terms involved in the functional $\mathcal{F}_\lambda$ act on these test functions. At this point, the application of the extension technique to carry out these tasks becomes unavoidable. Hence, trough this subsection we will focus on proving the PS condition for the extension functionals $\mathcal{H}_\lambda^\beta$ and $\mathcal{H}_\lambda^{\alpha,\beta}$. Once we have completed this task, since the β-extension is an isometry, given a PS sequence $\{(w_n, z_n)\} \subset \mathcal{X}_0^\beta(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$ at level c for the functional $\mathcal{H}_\lambda^\beta$ satisfying the PS condition it is clear that the sequence $\{(u_n, v_n)\} = \{Tr[w_n], Tr[z_n]\}$ belongs to $H_0^\beta(\Omega) \times H_0^\beta(\Omega)$ and it is a PS sequence at level c for the functional $\mathcal{J}_\lambda^\beta$ satisfying the PS condition and, thus, the functional $\mathcal{J}_\lambda^\beta$ satisfies the PS condition at every level c below certain critical level c^*. In a similar way we can infer that the functional $\mathcal{J}_\lambda^{\alpha,\beta}$ satisfies the PS condition.

More specifically, by means of a concentration-compactness argument relying on [**18**, Theorem 5.1] we first prove that the PS condition is satisfied for any level c such that

$$c < c^*_\beta = \left(\frac{1}{2} - \frac{1}{2^*_\beta}\right) (\kappa_\beta S(\beta, N))^{\frac{2^*_\beta}{2^*_\beta - 2}} = \frac{\beta}{N} (\kappa_\beta S(\beta, N))^{\frac{N}{2\beta}},$$

when dealing with the functional $\mathcal{H}^\beta_\lambda$, and for any level

$$c < c^*_\mu = \frac{1}{\lambda^{1-\beta/\alpha}} \left(\frac{1}{2} - \frac{1}{2^*_\mu}\right) (\kappa_\mu S(\mu, N))^{\frac{2^*_\mu}{2^*_\mu - 2}} = \frac{1}{\lambda^{1-\beta/\alpha}} \frac{\mu}{N} (\kappa_\mu S(\mu, N))^{\frac{N}{2\mu}},$$

when dealing with the functional $\mathcal{H}^{\alpha,\beta}_\lambda$. Secondly, using an appropriate cut-off version of the extremal functions (5.1.9) of the Sobolev inequality we will obtain a sequence below the critical levels c^*_β and c^*_μ.

5.3.1. Palais-Smale condition under the critical level.

We begin by proving that the PS condition is satisfied at levels c below certain critical level c^* to be determined next. To accomplish this first step, let us start recalling the following.

DEFINITION 5.3.1. *We said that a sequence $\{y^{1-2\mu}|\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight if for any $\eta > 0$ there exists $\rho_0 > 0$ such that*

$$\int_{\{y>\rho_0\}} \int_\Omega y^{1-2\mu} |\nabla w_n|^2 dxdy \leq \eta, \quad \forall n \in \mathbb{N}.$$

In particular, since we are dealing with a system, we said that the sequence

$$\{(y^{1-2\mu}|\nabla w_n|^2, y^{1-2\beta}|\nabla z_n|^2)\}_{n\in\mathbb{N}},$$

is tight if for any $\eta > 0$ there exists $\rho_0 > 0$ such that

$$\int_{\{y>\rho_0\}} \int_\Omega y^{1-2\mu} |\nabla w_n|^2 dxdy + \int_{\{y>\rho_0\}} \int_\Omega y^{1-2\beta} |\nabla z_n|^2 dxdy \leq \eta, \quad \forall n \in \mathbb{N}.$$

Now we state the Concentration-Compactness Theorem [**18**, Theorem 5.1] that will be the core of the proof of the PS condition.

THEOREM 5.3.1. *Let $\{w_n\}$ be a weakly convergent sequence to w in $\mathcal{X}^\mu_0(\mathscr{C}_\Omega)$ such that the sequence $\{y^{1-2\mu}|\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight. Let $u_n = w_n(x, 0)$ and $u = w(x, 0)$. Let ν, ζ be two nonnegative measures such that*

$$y^{1-2\mu}|\nabla w_n|^2 \to \zeta \quad \text{and} \quad |u_n|^{2^*_\mu} \to \nu, \quad \text{as } n \to \infty$$

in the sense of measures. Then there exists a set I, which is at most countable, containing the points $\{x_i\}_{i\in I} \subset \Omega$ and two positive numbers ν_i, ζ_i, with $i \in I$, such that,

- $\nu = |u|^{2^*_\mu} + \sum\limits_{i\in I} \nu_i \delta_{x_i},\ \nu_i > 0,$
- $\zeta = y^{1-2\mu}|\nabla w|^2 + \sum\limits_{i\in I} \zeta_i \delta_{x_i},\ \zeta_i > 0,$

where δ_{x_j} stands for the Dirac's delta centered at x_j and satisfying the condition

$$\zeta_i \geq S(\mu, N) \nu_i^{2/2^*_\mu}.$$

We are now ready to complete the first step of our argument.

LEMMA 5.3.1. *If $p = 2_\beta^* - 1$ the functional $\mathcal{H}_\lambda^\beta$ satisfies the Palais-Smale condition for any level c below the critical level c_β^*.*

PROOF. Let $\{(w_n, z_n)\}_{n\in\mathbb{N}} \subset \mathcal{X}_0^\beta(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$ be a PS sequence at level c for the functional $\mathcal{H}_\lambda^\beta$, i.e.

$$\mathcal{H}_\lambda^\beta(w_n, z_n) \to c < c_\beta^* \quad \text{and} \quad \left(\mathcal{H}_\lambda^\beta\right)'(w_n, z_n) \to 0. \tag{5.3.1}$$

From (5.3.1) and Lemma 5.2.7 we get that the sequence $\{(w_n, z_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{X}_0^\beta(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$, in other words, there exists a finite $M > 0$ such that

$$||w_n||^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)} + ||z_n||^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)} \leq M, \tag{5.3.2}$$

and, as a consequence, we can assume that, up to a subsequence,

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{weakly in } \mathcal{X}_0^\beta(\mathscr{C}_\Omega),\\ &w_n(x,0) \to w(x,0) \quad \text{strong in } L^r(\Omega), \text{with} \quad 1 \leq r < 2_\beta^*,\\ &w_n(x,0) \to w(x,0) \quad \text{a.e. in } \Omega, \end{aligned} \tag{5.3.3}$$

and

$$\begin{aligned} &z_n \rightharpoonup z \quad \text{weakly in } \mathcal{X}_0^\beta(\mathscr{C}_\Omega),\\ &z_n(x,0) \to z(x,0) \quad \text{strong in } L^r(\Omega), 1 \leq r < 2_\beta^*,\\ &z_n(x,0) \to z(x,0) \quad \text{a.e. in } \Omega. \end{aligned} \tag{5.3.4}$$

Before applying Theorem 5.3.1, first we need to check that the PS sequence $\{(w_n, z_n)\}_{n\in\mathbb{N}}$ is tight. To avoid any unnecessary technical details, and since the functional $\mathcal{H}_\lambda^\beta$ is obtained as a particular case (up to a multiplication by $\sqrt{\lambda}$) of the functional $\mathcal{H}_\lambda^{\alpha,\beta}$ we prove the following.

LEMMA 5.3.2. *A PS sequence $\{(w_n, z_n)\}_{n\in\mathbb{N}} \subset \mathcal{X}_0^\mu(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$ at level c for the functional $\mathcal{H}_\lambda^{\alpha,\beta}$ is tight.*

PROOF. The proof is similar to the proof of Lemma 3.6 in [**18**], which follows some arguments contained in [**11**], and we include it for the reader's convenience. By contradiction, suppose that there exists $\eta_0 > 0$ and $m_0 \in \mathbb{N}$ such that for any $\rho > 0$ we have, up to a subsequence,

$$\int_{\{y>\rho\}}\int_\Omega y^{1-2\mu}|\nabla w_n|^2 dxdy + \int_{\{y>\rho\}}\int_\Omega y^{1-2\beta}|\nabla z_n|^2 dxdy > \eta_0, \quad \forall m \geq m_0. \tag{5.3.5}$$

Let $\varepsilon > 0$ be fixed (to be determined later), and let $\rho_0 > 0$ such that

$$\int_{\{y>\rho_0\}}\int_\Omega y^{1-2\mu}|\nabla w|^2 dxdy + \int_{\{y>\rho_0\}}\int_\Omega y^{1-2\beta}|\nabla z|^2 dxdy < \varepsilon.$$

Let $j=\left[\frac{M}{\varepsilon\kappa}\right]$ be the integer part, with $\kappa=\min\{\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}},\frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\}$ and let us define the sets $I_k=\{y\in\mathbb{R}^+:\rho_0+k\leq y\leq \rho_0+k+1\}$, $k=0,1,\ldots,j$. Then, using (5.3.2)

$$\begin{aligned}\sum_{k=0}^{j}\int_{I_k}\int_\Omega y^{1-2\mu}|\nabla w_n|^2dxdy+\int_{I_k}\int_\Omega y^{1-2\beta}|\nabla z_n|^2dxdy&\\ \leq \int_{\mathscr{C}_\Omega}y^{1-2\mu}|\nabla w_n|^2dxdy&+\int_{\mathscr{C}_\Omega}y^{1-2\beta}|\nabla z_n|^2dxdy\\ \leq\frac{M}{\kappa}<\varepsilon(j+1).&\end{aligned}$$

Hence, there exists $k_0\in\{0,1,\ldots,j\}$ such that

$$\int_{I_{k_0}}\int_\Omega y^{1-2\mu}|\nabla w_n|^2dxdy+\int_{I_{k_0}}\int_\Omega y^{1-2\beta}|\nabla z_n|^2dxdy\leq\varepsilon. \tag{5.3.6}$$

Take now a regular cut-off function

$$X(y)=\begin{cases}0 & \text{if } y\leq r+k_0,\\ 1 & \text{if } y\geq r+k_0+1,\end{cases}$$

and define $(t_n,s_n)=(X(y)w_n,X(y)z_n)$. Then

$$\begin{aligned}&\left|\left\langle\left(\mathcal{H}_\lambda^{\alpha,\beta}\right)'(w_n,z_n)-\left(\mathcal{H}_\lambda^{\alpha,\beta}\right)'(t_n,s_n)|(t_n,s_n)\right\rangle\right|\\ &=\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}}\int_{\mathscr{C}_\Omega}y^{1-2\mu}\langle\nabla(w_n-t_n),\nabla t_n\rangle dxdy+\frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\int_{\mathscr{C}_\Omega}y^{1-2\beta}\langle\nabla(z_n-s_n),\nabla s_n\rangle dxdy\\ &=\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}}\int_{I_{k_0}}\int_\Omega y^{1-2\mu}\langle\nabla(w_n-t_n),\nabla t_n\rangle dxdy+\frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\int_{I_{k_0}}\int_\Omega y^{1-2\beta}\langle\nabla(z_n-s_n),\nabla s_n\rangle dxdy.\end{aligned}$$

Let us set $I_{k_0}^*=I_{k_0}\times\Omega$. Because of the Cauchy-Schwarz inequality, inequality (5.3.6) and the compact inclusion[1],

$$H^1(I_{k_0}^*,y^{1-2\mu}dxdy)\times H^1(I_{k_0}^*,y^{1-2\beta}dxdy)\hookrightarrow L^2(I_{k_0}^*,y^{1-2\mu}dxdy)\times L^2(I_{k_0}^*,y^{1-2\beta}dxdy),$$

provided by [**48**, Theorem 1.2], it follows that,

$$\begin{aligned}&\left|\left\langle\left(\mathcal{H}_\lambda^{\alpha,\beta}\right)'(w_n,z_n)-\left(\mathcal{H}_\lambda^{\alpha,\beta}\right)'(t_n,s_n)|(t_n,s_n)\right\rangle\right|\\ &\leq\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}}\left(\int_{I_{k_0}}\int_\Omega y^{1-2\mu}|\nabla(w_n-t_n)|^2dxdy\right)^{1/2}\left(\int_{I_{k_0}}\int_\Omega y^{1-2\mu}|\nabla t_n|^2dxdy\right)^{1/2}\\ &+\frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\left(\int_{I_{k_0}}\int_\Omega y^{1-2\beta}|\nabla(z_n-s_n)|^2dxdy\right)^{1/2}\left(\int_{I_{k_0}}\int_\Omega y^{1-2\beta}|\nabla s_n|^2dxdy\right)^{1/2}\\ &\leq\max\left\{\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}},\frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\right\}c\varepsilon\leq C\varepsilon,\end{aligned}$$

[1] Let us recall that $\beta\in(0,1)$ and $\mu:=\alpha-\beta\in(0,1)$ thus, the weights $w_1(x,y)=y^{1-2\mu}$ and $w_2(x,y)=y^{1-2\beta}$ belongs to the Muckenhoupt class A_2. We refer to [**48**] for the precise definition as well as some useful properties of the weights belonging to the Muckenhoupt classes A_p.

where $C := c\max\left\{\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}}, \frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\right\}$. On the other hand, by (5.3.1)

$$\left|\left\langle \left(\mathcal{H}_\lambda^{\alpha,\beta}\right)'(t_n,s_n)|(t_n,s_n)\right\rangle\right| \leq c_1\varepsilon + o(1),$$

with c_1 a positive constant. Thus, we conclude

$$\begin{aligned}\int_{\{y>r+k_0+1\}}\int_\Omega y^{1-2\mu}|\nabla w_n|^2dxdy + \int_{\{y>r+k_0+1\}}\int_\Omega y^{1-2\beta}|\nabla z_n|^2dxdy&\\ \leq \int_{\mathscr{C}_\Omega} y^{1-2\mu}|\nabla t_n|^2dxdy + \int_{\mathscr{C}_\Omega} y^{1-2\beta}|\nabla s_n|^2dxdy&\\ \leq \frac{1}{\kappa}\left\langle \left(\mathcal{H}_\lambda^{\alpha,\beta}\right)'(t_n,s_n)|(t_n,s_n)\right\rangle \leq C\varepsilon,&\end{aligned}$$

in contradiction with (5.3.5). Hence, the sequence is tight. $\square$

Continuation proof Lemma 5.3.1. Once we have proved that the PS sequence

$$\{(w_n,z_n)\}_{n\in\mathbb{N}} \subset \mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega),$$

is tight, we can apply Theorem 5.3.1. Consequently, up to a subsequence, there exists an index I, at most countable, a sequence of points $\{x_k\}\subset\Omega$ and non-negative real numbers ν_k and ζ_k such that

- $|u_n|^{2_\beta^*} \to \nu = |u|^{2_\beta^*} + \sum\limits_{i\in I}\nu_i\delta_{x_i}$,
- $y^{1-2\beta}|\nabla w_n|^2 \to \zeta = y^{1-2\beta}|\nabla w|^2 + \sum\limits_{i\in I}\zeta_i\delta_{x_i}$,
- $y^{1-2\beta}|\nabla z_n|^2 \to \widetilde{\zeta} = y^{1-2\beta}|\nabla z|^2 + \sum\limits_{i\in I}\widetilde{\zeta}_i\delta_{x_i}$,

where δ_{x_j} is the Dirac's delta centered at x_j and satisfying,

$$\zeta_i \geq S(\mu,N)\nu_i^{2/2_\mu^*}. \tag{5.3.7}$$

in the sense of measures. We fix $j\in I$ and we let $\phi\in\mathcal{C}_0^\infty(\mathbb{R}_+^{N+1})$ be a non-increasing cut-off function verifying $\phi=1$ in $B_1^+(x_j)$, $\phi=0$ in $B_2^+(x_j)^c$, with $B_r^+(x_j)\subset\mathbb{R}^N\times\{y\geq 0\}$ the $(N+1)$-dimensional semi-ball of radius $r>0$ centered at x_j. Let now $\phi_\varepsilon(x,y)=\phi(x/\varepsilon,y/\varepsilon)$, such that $|\nabla\phi_\varepsilon|\leq\frac{C}{\varepsilon}$ and denote $\Gamma_{2\varepsilon}=B_{2\varepsilon}^+(x_j)\cap\{y=0\}$. Therefore, since by (5.3.1)

$$\left(\mathcal{H}_\lambda^\beta\right)'(w_n,z_n)\to 0 \quad \text{in the dual space} \quad \left(\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\right)', \tag{5.3.8}$$

using $(\phi_\varepsilon w_n,\phi_\varepsilon z_n)$ as a test function in (5.3.8), we obtain

$$\begin{aligned}\lim_{n\to\infty}\bigg(\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\nabla w_n\nabla(\phi_\varepsilon w_n)dxdy + \kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\nabla z_n\nabla(\phi_\varepsilon z_n)dxdy&\\ -2\sqrt{\lambda}\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon w_n(x,0)z_n(x,0)dx - \int_{\Gamma_{2\varepsilon}}\phi_\varepsilon|w_n|^{2_\beta^*}(x,0)dx\bigg) = 0.&\end{aligned}$$

Hence,

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla w_n,\nabla\phi_\varepsilon\rangle w_n dxdy+\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla z_n,\nabla\phi_\varepsilon\rangle z_n dxdy\right)\\
&=\lim_{n\to\infty}\left(2\sqrt{\lambda}\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon w_n(x,0)z_n(x,0)dx+\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon|w_n|^{2_\beta^*}(x,0)dx\right.\\
&\qquad\left.-\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\phi_\varepsilon|\nabla w_n|^2dxdy-\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\phi_\varepsilon|\nabla z_n|^2dxdy\right).
\end{aligned}
$$

Moreover, thanks to (5.3.3), (5.3.4) and Theorem 5.3.1, we find,

(5.3.9)
$$
\begin{aligned}
&\lim_{n\to\infty}\left(\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla w_n,\nabla\phi_\varepsilon\rangle w_n dxdy+\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla z_n,\nabla\phi_\varepsilon\rangle z_n dxdy\right)\\
&\quad=2\sqrt{\lambda}\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon w(x,0)z(x,0)dx+\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon d\nu-\kappa_\beta\int_{B_{2\varepsilon}^+(x_j)}\phi_\varepsilon d\zeta-\kappa_\beta\int_{B_{2\varepsilon}^+(x_j)}\phi_\varepsilon d\widetilde{\zeta}.
\end{aligned}
$$

Assume for the moment that the left hand side of (5.3.9) vanishes as $\varepsilon\to 0$. Then, it follows that,

$$
\begin{aligned}
0&=\lim_{\varepsilon\to 0}2\sqrt{\lambda}\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon w(x,0)z(x,0)dx+\int_{\Gamma_{2\varepsilon}}\phi_\varepsilon d\nu-\kappa_\beta\int_{B_{2\varepsilon}^+(x_j)}\phi_\varepsilon d\zeta-\kappa_\beta\int_{B_{2\varepsilon}^+(x_j)}\phi_\varepsilon d\widetilde{\zeta}\\
&=\nu_j-\kappa_\beta\zeta_j-\kappa_\beta\widetilde{\zeta}_j,
\end{aligned}
$$

and we conclude,

$$
\nu_j=\kappa_\beta\left(\zeta_j+\widetilde{\zeta}_j\right). \tag{5.3.10}
$$

Finally, we have two options, either the compactness of the PS sequence or concentration around those points x_j. In other words, either $\nu_j=0$, so that $\zeta_j=\widetilde{\zeta}_j=0$ or, thanks to (5.3.10) and (5.3.7), $\nu_j\geq(\kappa_\beta S(\beta,N))^{\frac{2_\beta^*}{2_\beta^*-2}}$. In case of having concentration, we find,

$$
\begin{aligned}
c&=\lim_{n\to\infty}\mathcal{H}_\lambda^\beta(w_n,z_n)=\lim_{n\to\infty}\mathcal{H}_\lambda^\beta(w_n,z_n)-\frac{1}{2}\left\langle\left(\mathcal{H}_\lambda^\beta\right)(w_n,z_n)|(w_n,z_n)\right\rangle\\
&=\left(\frac{1}{2}-\frac{1}{2_\beta^*}\right)\int_\Omega|w(x,0)|^{2_\beta^*}dx+\left(\frac{1}{2}-\frac{1}{2_\beta^*}\right)\nu_{k_0}\\
&\geq\left(\frac{1}{2}-\frac{1}{2_\beta^*}\right)(\kappa_\beta S(\beta,N))^{\frac{2_\beta^*}{2_\beta^*-2}}=c_\beta^*,
\end{aligned}
$$

in contradiction with the hypotheses $c<c_\beta^*$. It only remains to prove that the left hand side of (5.3.9) vanishes as $\varepsilon\to 0$. Due to (5.3.1) and Lemma 5.2.7, the PS sequence $\{(w_n,z_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega)$, so that, up to a subsequence,

$$
\begin{aligned}
&(w_n,z_n)\rightarrow(w,z)\in\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega),\\
&(w_n,z_n)\rightarrow(w,z)\quad\text{a.e. in}\quad\mathscr{C}_\Omega.
\end{aligned}
$$

Moreover, for $r < 2^* = \frac{2(N+1)}{N-1}$ we have the compact inclusion, [**48**, Theorem 1.2],

$$\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\hookrightarrow L^r(\mathscr{C}_\Omega,y^{1-2\beta}dxdy)\times L^r(\mathscr{C}_\Omega,y^{1-2\beta}dxdy).$$

Applying Hölder's inequality with $p=\frac{N+1}{N-1}$ and $q=\frac{N+1}{2}$, we find,

$$\begin{aligned}
&\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|\nabla\phi_\varepsilon|^2|w_n|^2dxdy\\
&\leq\left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|\nabla\phi_\varepsilon|^{N+1}dxdy\right)^{\frac{2}{N+1}}\left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|w_n|^{2\frac{N+1}{N-1}}dxdy\right)^{\frac{N-1}{(N+1)}}\\
&\leq\frac{1}{\varepsilon^2}\left(\int_{B_{2\varepsilon}(x_{k_0})}\int_0^\varepsilon y^{1-2\beta}dxdy\right)^{\frac{2}{N+1}}\left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|w_n|^{2\frac{N+1}{N-1}}dxdy\right)^{\frac{N-1}{(N+1)}}\\
&\leq c_0\varepsilon^{\frac{2(1-2\beta)}{N+1}}\left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|w_n|^{2\frac{N+1}{N-1}}dxdy\right)^{\frac{N-1}{(N+1)}}\\
&\leq c_0\varepsilon^{\frac{2(1-2\beta)}{N+1}}\varepsilon^{\frac{(2+N-2\beta)(N-1)}{(N+1)}}\left(\int_{B_2^+(x_{k_0})} y^{1-2\beta}|w_n(\varepsilon x,\varepsilon y)|^{2\frac{N+1}{N-1}}dxdy\right)^{\frac{N-1}{(N+1)}}\\
&\leq c_1\varepsilon^{N-2\beta}.
\end{aligned}$$

for appropriate constants c_0 and c_1. In a similar way,

$$\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|\nabla\phi_\varepsilon|^2|z_n|^2dxdy\leq c_2\varepsilon^{N-2\beta}.$$

Thus, we find that,

$$\begin{aligned}
0\leq\lim_{n\to\infty}&\left|\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla w_n,\nabla\phi_\varepsilon\rangle w_ndxdy+\kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla z_n,\nabla\phi_\varepsilon\rangle z_ndxdy\right|\\
&\leq\kappa_\beta\lim_{n\to\infty}\left(\int_{\mathscr{C}_\Omega} y^{1-2\beta}|\nabla w_n|^2dxdy\right)^{1/2}\left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2s}|\nabla\phi_\varepsilon|^2|w_n|^2dxdy\right)^{1/2}\\
&+\kappa_\beta\lim_{n\to\infty}\left(\int_{\mathscr{C}_\Omega} y^{1-2\beta}|\nabla z_n|^2dxdy\right)^{1/2}\left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2s}|\nabla\phi_\varepsilon|^2|z_n|^2dxdy\right)^{1/2}\\
&\leq C\varepsilon^{\frac{N-2\beta}{2}}\to 0,
\end{aligned}$$

as $\varepsilon\to 0$ and the proof of the lemma is complete. □

Next we show the corresponding result for the functional $\mathcal{H}_\lambda^{\alpha,\beta}$.

LEMMA 5.3.3. *If $p=2_\mu^*-1$ the functional $\mathcal{H}_\lambda^{\alpha,\beta}$ satisfies the Palais-Smale condition for any level c below the critical level c_μ^*.*

PROOF. Proceeding as in the proof of Lemma 5.3.1 we find that,

$$0=\frac{1}{\lambda^{1-\beta/\alpha}}\nu_j-\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}}\zeta_j-\frac{\kappa_\beta}{\lambda^{\beta/\alpha}}\widetilde{\zeta}_j, \tag{5.3.11}$$

for the measures given by the concentration-compactness argument and, thus, we conclude

$$\nu_j = \kappa_\mu \zeta_j + \lambda^{1-2\beta/\alpha}\kappa_\beta \widetilde{\zeta}_j. \tag{5.3.12}$$

We have, once again, two options, either the compactness of the PS sequence or concentration around those points x_j. In other words, either $\nu_j = 0$, so that $\zeta_j = \widetilde{\zeta}_j = 0$ or, thanks to (5.3.12) and (5.3.7), $\nu_j \geq (\kappa_\mu S(\mu, N))^{\frac{2_\mu^*}{2_\mu^*-2}}$. In case of having concentration, we find that,

$$\begin{aligned} c = \lim_{n\to\infty} \mathcal{H}_\lambda^{\alpha,\beta}(w_n, z_n) &= \lim_{n\to\infty} \mathcal{H}_\lambda^{\alpha,\beta}(w_n, z_n) - \frac{1}{2}\left\langle \left(\mathcal{H}_\lambda^{\alpha,\beta}\right)(w_n, z_n)|(w_n, z_n)\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_\mu^*}\right)\frac{1}{\lambda^{1-\beta/\alpha}}\int_\Omega |w(x,0)|^{2_\mu^*}dx + \left(\frac{1}{2} - \frac{1}{2_\mu^*}\right)\frac{1}{\lambda^{1-\beta/\alpha}}\nu_{k_0} \\ &\geq \left(\frac{1}{2} - \frac{1}{2_\mu^*}\right)\frac{1}{\lambda^{1-\beta/\alpha}}\left(\kappa_\mu S(\mu, N)\right)^{\frac{2_\mu^*}{2_\mu^*-2}} = c_\mu^*, \end{aligned}$$

in contradiction with the hypotheses $c < c_\mu^*$. As performed in the proof of Lemma 5.3.1, to obtain equality (5.3.11) we have to prove that, given a PS sequence $\{(w_n, z_n)\}_{n\in\mathbb{N}}$ at level c for the functional $\mathcal{H}_\lambda^{\alpha,\beta}$, it is satisfied that,

$$\lim_{n\to\infty}\left(\kappa_\mu \int_{\mathscr{C}_\Omega} y^{1-2\mu}\langle\nabla w_n, \nabla\phi_\varepsilon\rangle w_n dxdy + \kappa_\beta\int_{\mathscr{C}_\Omega} y^{1-2\beta}\langle\nabla z_n, \nabla\phi_\varepsilon\rangle z_n dxdy\right) \to 0, \quad \text{as } \varepsilon \to 0.$$

Moreover, since the inclusion,

$$\mathcal{X}_0^\mu(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega) \hookrightarrow L^r(\mathscr{C}_\Omega, y^{1-2\mu}dxdy) \times L^r(\mathscr{C}_\Omega, y^{1-2\beta}dxdy),$$

is compact for $r < 2^* = \frac{2(N+1)}{N-1}$, arguing as in the proof of Lemma 5.3.1 we find that,

$$\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\mu}|\nabla\phi_\varepsilon|^2|w_n|^2 dxdy \leq c_1\varepsilon^{N-2\mu},$$

as well as,

$$\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-2\beta}|\nabla\phi_\varepsilon|^2|z_n|^2 dxdy \leq c_2\varepsilon^{N-2\beta},$$

and we conclude as in the proof of Lemma 5.3.1. □

5.3.2. PS sequences under the critical level.

At this point, it remains to show that we can obtain PS sequences for the functionals $\mathcal{H}_\lambda^\beta$ and $\mathcal{H}_\lambda^{\alpha,\beta}$ under the critical levels c_β^* and c_μ^* respectively. In order to get such sequences, we consider the extremal functions of the fractional Sobolev inequality (5.1.8), namely, given $\theta \in (0, 1)$, we set

$$u_\varepsilon^\theta(x) = \frac{\varepsilon^{\frac{N-2\theta}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2\theta}{2}}},$$

and $w_\varepsilon^\theta = E_\theta[u_\varepsilon^\theta]$ its θ-extension function. Note that both functions u_ε^θ and the Poisson kernel (1.2.3) are self-similar functions, $u_\varepsilon^\theta(x) = \varepsilon^{-\frac{N-2\theta}{2}}u_1(x)$, and $P_y^s(x) = \frac{1}{y^N}P_1^s\left(\frac{x}{y}\right)$ so the extension family $w_\varepsilon^\theta = E_s[u_\varepsilon^\theta]$ satisfies

$$w_\varepsilon^\theta(x) = \varepsilon^{-\frac{N-2\theta}{2}}w_1^\theta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right). \tag{5.3.13}$$

Then, since w_ε^θ is a minimizer,

$$S(\theta,N)=\frac{\displaystyle\int_{\mathbb{R}_+^{N+1}} y^{1-2\theta}|\nabla w_\varepsilon^\theta|^2dxdy}{\|u_\varepsilon^\theta\|^2_{L^{2_\theta^*}(\mathbb{R}^N)}}.$$

Now consider a non-increasing smooth cut-off function $\phi_0(t)\in\mathcal{C}_0^\infty(\mathbb{R}_+)$ such that $\phi_0(t)=1$ if $0\le t\le 1/2$ and $\phi_0(t)=0$ if $s\ge 1$. Assume without loss of generality that $0\in\Omega$, and define, for some $r>0$ small enough such that $\overline{B}_r^+\subseteq\overline{\mathcal{C}}_\Omega$, the function $\phi_r(x,y)=\phi_0(\frac{r_{x,y}}{r})$ where $r_{xy}=|(x,y)|=\left(|x|^2+y^2\right)^{1/2}$. Note that $\phi_r w_\varepsilon^\theta\in\mathcal{X}_0^\theta(\mathscr{C}_\Omega)$. We recall now the following lemma proved in [**18**].

Lemma 5.3.4. *The family $\{\phi_r w_\varepsilon^\theta\}$ and its trace on $\{y=0\}$, denoted by $\{\phi_r u_\varepsilon^\theta\}$, satisfy*

$$\|\phi_r w_\varepsilon^\theta\|^2_{\mathcal{X}_0^\theta(\mathscr{C}_\Omega)}=\|w_\varepsilon^\theta\|^2_{\mathcal{X}_0^\theta(\mathscr{C}_\Omega)}+O(\varepsilon^{N-2\theta}),$$
$$\|\phi_r u_\varepsilon^\theta\|^2_{L^2(\Omega)}=\begin{cases} C\varepsilon^{2\theta}+O(\varepsilon^{N-2\theta}) & \text{if } N>4\theta,\\ C\varepsilon^{2\theta}|\log(\varepsilon)| & \text{if } N=4\theta.\end{cases}$$

Remark 5.3.1. *Since $\|u_\varepsilon^\theta\|_{L^{2_\theta^*}(\mathbb{R}^N)}\sim C$ does not depend on ε it follows that*

$$\|\phi_r u_\varepsilon^\theta\|_{L^{2_\theta^*}(\Omega)}=\|u_\varepsilon^\theta\|_{L^{2_\theta^*}(\mathbb{R}^N)}+O(\varepsilon^N)=C+O(\varepsilon^N).$$

To continue we consider the normalized functions,

$$\eta_\varepsilon^\theta=\frac{\phi_r w_\varepsilon^\theta}{\|\phi_r u_\varepsilon^\theta\|_{2_\theta^*}}\quad\text{and}\quad \sigma_\varepsilon^\theta=\frac{\phi_r u_\varepsilon^\theta}{\|\phi_r u_\varepsilon^\theta\|_{2_\theta^*}}.$$

Then, because of Lemma 5.3.4 the following estimates are satisfied,

$$\begin{aligned}&\|\eta_\varepsilon^\theta\|^2_{\mathcal{X}_0^\theta(\mathscr{C}_\Omega)}=S(\theta,N)+O(\varepsilon^{N-2\theta}),\\ &\|\sigma_\varepsilon^\theta\|^2_{L^2(\Omega)}=\begin{cases} C\varepsilon^{2\theta}+O(\varepsilon^{N-2\theta}) & \text{if } N>4\theta,\\ C\varepsilon^{2\theta}|\log(\varepsilon)| & \text{if } N=4\theta,\end{cases}\\ &\|\sigma_\varepsilon^\theta\|_{L^{2_\theta^*}(\Omega)}=1.\end{aligned}\tag{5.3.14}$$

To simplify the notation, in the sequel we will write $F(\varepsilon):=\|\sigma_\varepsilon^\theta\|^2_{L^2(\Omega)}$ when referring to the estimates (5.3.14), differentiating whether $N>4\theta$ or $N=4\theta$.
Now, for the functional $\mathcal{H}_\lambda^\beta$ consider the pair

$$(\overline{w}_\varepsilon^\beta,\overline{z}_\varepsilon^\beta)=(M\eta_\varepsilon^\beta,M\rho\eta_\varepsilon^\beta),\tag{5.3.15}$$

with $\rho>0$ to be determined and $M>0$ a sufficiently large constant such that $\mathcal{H}_\lambda^\beta(\overline{w}_\varepsilon^\beta,\overline{z}_\varepsilon^\beta)<0$. Then, under this construction, we define the paths

$$\Gamma_\varepsilon:=\{\gamma\in C([0,1],\mathcal{X}_0^\beta(\mathscr{C}_\Omega)\times\mathcal{X}_0^\beta(\mathscr{C}_\Omega))\,;\,\gamma(0)=(0,0),\ \gamma(1)=(\overline{w}_\varepsilon^\beta,\overline{z}_\varepsilon^\beta)\},$$

and we consider the minimax values

$$c_\varepsilon=\inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]}\mathcal{H}_\lambda^\beta(\gamma(t)).$$

To complete the second step of our argument we prove that, in fact, $c_\varepsilon<c_\beta^*$ for ε small enough.

LEMMA 5.3.5. *Assume $p = 2_\beta^* - 1$. Then, there exists $\varepsilon > 0$ small enough such that,*

$$\sup_{t\geq 0} \mathcal{H}_\lambda^\beta(t\overline{w}_\varepsilon^\beta, t\overline{z}_\varepsilon^\beta) < c_\beta^*, \tag{5.3.16}$$

provided that $N > 6\beta$.

PROOF. Using the test functions (5.3.15) and applying the estimates (5.3.14) with $\theta = \beta$, it follows that

$$\begin{aligned} g(t) :=&\mathcal{H}_\lambda^\beta(t\overline{w}_\varepsilon^\beta, t\overline{z}_\varepsilon^\beta) \\ =&\frac{M^2t^2}{2}\left(\kappa_\beta\|\eta_\varepsilon^\beta\|^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)} + \rho^2\kappa_\beta\|\eta_\varepsilon^\beta\|^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)} - 2\sqrt{\lambda}\|\sigma_\varepsilon^\theta\|^2_{L^2(\Omega)}\right) - \frac{Mt^{2_\beta^*}}{2_\beta^*} \\ =&\frac{M^2t^2}{2}\left([\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] + \rho^2[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] - 2\sqrt{\lambda}F(\varepsilon)\right) \\ &-\frac{M^{2_\beta^*}t^{2_\beta^*}}{2_\beta^*}. \end{aligned}$$

It is clear that $\lim\limits_{t\to\infty} g(t) = -\infty$, therefore, the function $g(t)$ possesses a maximum value at the point,

$$t_\varepsilon := \left(\frac{M^2\left([\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] + \rho^2[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] - 2\sqrt{\lambda}F(\varepsilon)\right)}{M^{2_\beta^*}}\right)^{\frac{1}{2_\beta^*-2}}.$$

Moreover, at this point $t_{\lambda,\varepsilon}$ we have,

$$g(t_\varepsilon) = \left(\frac{1}{2} - \frac{1}{2_\beta^*}\right)\left([\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] + \rho^2[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] - 2\sqrt{\lambda}F(\varepsilon)\right)^{\frac{2_\beta^*}{2_\beta^*-2}}.$$

We now note that the proof will be completed if the inequality,

$$g(t_\varepsilon) < \left(\frac{1}{2} - \frac{1}{2_\beta^*}\right)(\kappa_\beta S(\beta,N))^{\frac{2_\beta^*}{2_\beta^*-2}} = c_\beta^*, \tag{5.3.17}$$

holds true for ε small enough and making the appropriate election of $\rho > 0$. Thus, simplifying (5.3.17), we are left to choose $\rho > 0$ such that

$$O(\varepsilon^{N-2\beta}) + \kappa_\beta S(\beta,N)\rho^2 + O(\varepsilon^{N-2\beta})\rho^2 < 2\sqrt{\lambda}\rho F(\varepsilon),$$

holds true provided that ε is small enough. To this end, take $\rho = \varepsilon^\delta$ with $\delta > 0$ to be determined, therefore, since

$$O(\varepsilon^{N-2\beta}) + \kappa_\beta S(\beta,N)\varepsilon^{2\delta} + O(\varepsilon^{N-2\beta+2\delta}) = O(\varepsilon^\tau),$$

with $\tau = \min\{N-2\beta, 2\delta, N-2\beta+2\delta\} = \min\{N-2\beta, 2\delta\}$, the proof will be finished once $\delta > 0$ has been chosen such that the inequality

$$O(\varepsilon^\tau) < 2\sqrt{\lambda}\rho F(\varepsilon), \tag{5.3.18}$$

holds true for $\varepsilon > 0$ small enough. Now we use the estimates (5.3.14). Then, if $N = 4\beta$ inequality (5.3.18) reads

$$O(\varepsilon^\tau) < 2C\sqrt{\lambda}\varepsilon^{2\beta+\delta}|\log(\varepsilon)|. \tag{5.3.19}$$

Since $\varepsilon << 1$, inequality (5.3.19) holds if and only if $\tau = \min\{2\beta, 2\delta\} > 2\beta + \delta$, that is obviously impossible and, thus, inequality (5.3.18) can not hold for $N = 4\beta$. On the other hand, if $N > 4\beta$ inequality (5.3.18) has the form,

$$O(\varepsilon^{\tau}) < 2C\sqrt{\lambda}\varepsilon^{2\beta+\delta}. \tag{5.3.20}$$

Since $\varepsilon << 1$, inequality (5.3.20) holds if and only if $\tau = \min\{N-2\beta, 2\delta\} > 2\beta + \delta$. Using the identity $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$, we arrive at the condition

$$N - 2\beta - |N - 2\beta - 2\delta| > 4\beta. \tag{5.3.21}$$

Finally, we have two options,

(1) $N - 2\beta > 2\delta$ combined with (5.3.21) provides us with the range,

$$N - 2\beta > 2\delta > 4\beta. \tag{5.3.22}$$

Then $N > 6\beta$ necessarily, so that we can choose a positive δ satisfying (5.3.22) and, hence, inequality (5.3.18) holds for $\varepsilon > 0$ small enough.

(2) $N - 2\beta < 2\delta$ combined with (5.3.21) implies that $2(N-2\beta) - 4\beta > 2\delta$, and hence,

$$2(N-2\beta) - 4\beta > 2\delta > N - 2\beta, \tag{5.3.23}$$

Once again $N > 6\beta$ necessarily, so that we can choose a positive δ satisfying (5.3.23) and, hence, inequality (5.3.18) holds for $\varepsilon > 0$ small enough.

Thus, if $N > 6\beta$ we can choose $\rho > 0$ and $\varepsilon > 0$ small enough such that (5.3.16) is achieved. $\square$

Now, we are in the position to conclude the proof of the second main result of this chapter. First we will focus on the particular case when $\alpha = 2\beta$. Later on we will follow a similar argument to prove the results when $\alpha \neq 2\beta$.

Proof of Theorem 5.1.2. Case $\alpha = 2\beta$. Due to Lemma 5.3.5, we obtain,

$$c_\varepsilon \leq \sup_{t\geq 0} \mathcal{H}_\lambda^\beta(t\overline{w}_\varepsilon^\beta, t\overline{z}_\varepsilon^\beta) < c_\beta^*$$

for ε small enough and because of Lemma 5.3.1 the functional $\mathcal{H}_\lambda^\beta$ satisfies the PS condition for any level c_ε with ε small enough. Moreover, by Lemma 5.2.3, the functional $\mathcal{H}_\lambda^\beta$ has the mountain pass geometry. Therefore, we can apply the Mountain Pass Theorem to obtain the existence of a critical point $(w,z) \in \mathcal{X}_0^\beta(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$. The rest follows as in the subcritical case. $\square$

Now, we focus on the functional $\mathcal{H}_\lambda^{\alpha,\beta}$. For this case, we consider the pair

$$(\overline{w}_\varepsilon^\mu, \overline{z}_\varepsilon^\beta) = (M\eta_\varepsilon^\mu, M\rho\eta_\varepsilon^\beta), \tag{5.3.24}$$

with $\rho > 0$ to be determined and $M > 0$ a sufficiently large constant such that it is satisfied $\mathcal{H}_\lambda^{\alpha,\beta}(\overline{w}_\varepsilon^\mu, \overline{z}_\varepsilon^\beta) < 0$. Let us notice that, by definition,

$$\sigma_\varepsilon^\mu \sigma_\varepsilon^\beta = \frac{\phi_r u_\varepsilon^\mu \phi_r u_\varepsilon^\beta}{\|\phi_r u_\varepsilon^\mu\|_{2_\mu^*} \|\phi_r u_\varepsilon^\beta\|_{2_\beta^*}} = C\phi_r^2 u_\varepsilon^\mu u_\varepsilon^\beta,$$

and, since $\mu := \alpha - \beta$, we find

$$u_\varepsilon^\mu u_\varepsilon^\beta = \frac{\varepsilon^{\frac{N-2\mu}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2\mu}{2}}}\frac{\varepsilon^{\frac{N-2\beta}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2\beta}{2}}} = \frac{\varepsilon^{N-\alpha}}{(\varepsilon^2+|x|^2)^{N-\alpha}} = \left(\frac{\varepsilon^{\frac{N-2(\alpha/2)}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2(\alpha/2)}{2}}}\right)^2.$$

Thus, applying (5.3.14) with $\theta = \frac{\alpha}{2}$ it yields

$$\int_\Omega \sigma_\varepsilon^\mu \sigma_\varepsilon^\beta dx = C\|\phi_r u_\varepsilon^{\alpha/2}\|^2_{L^2(\Omega)} = \begin{cases} C\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ C\varepsilon^\alpha|\log(\varepsilon)| & \text{if } N = 2\alpha. \end{cases} \tag{5.3.25}$$

Following the steps performed for the case $\alpha = 2\beta$, we define the paths

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], \mathcal{X}_0^\mu(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega))\,;\, \gamma(0) = (0,0),\ \gamma(1) = (M\eta_\varepsilon^\mu, M\rho\eta_\varepsilon^\beta)\},$$

and we consider the minimax values

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} \mathcal{H}_\lambda^{\alpha,\beta}(\gamma(t)).$$

The final step of our scheme will be completed once we have shown that $c_\varepsilon < c_\mu^*$ for ε small enough.

LEMMA 5.3.6. *Assume $p = 2_\beta^* - 1$. Then, there exists $\varepsilon > 0$ small enough such that,*

$$\sup_{t\geq 0} \mathcal{H}_\lambda^{\alpha,\beta}(t\overline{w}_\varepsilon^\mu, t\overline{z}_\varepsilon^\beta) < c_\mu^*, \tag{5.3.26}$$

provided that $N > 4\alpha - 2\beta$.

PROOF. Thanks to (5.3.14) and using $F(\varepsilon)$ in this case in (5.3.25),

$$\begin{aligned} g(t) &:= \mathcal{H}_\lambda^{\alpha,\beta}(t\overline{w}_\varepsilon^\mu, t\overline{z}_\varepsilon^\beta) \\ &= \frac{M^2t^2}{2}\left(\frac{\kappa_\mu}{\lambda^{1-\beta/\alpha}}\|\eta_\varepsilon^\mu\|^2_{\mathcal{X}_0^\mu(\mathscr{C}_\Omega)} + \frac{\rho^2\kappa_\beta}{\lambda^{\beta/\alpha}}\|\eta_\varepsilon^\beta\|^2_{\mathcal{X}_0^\beta(\mathscr{C}_\Omega)} - 2\|\sigma_\varepsilon^{\alpha/2}\|^2_{L^2(\Omega)}\right) - \frac{M^{2_\mu^*}t^{2_\mu^*}}{2_\mu^*\lambda^{1-\beta/\alpha}} \\ &= \frac{M^2t^2}{2}\left(\frac{1}{\lambda^{1-\beta/\alpha}}[\kappa_\mu S(\mu,N) + O(\varepsilon^{N-2\mu})] + \frac{\rho^2}{\lambda^{\beta/\alpha}}[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] - 2F(\varepsilon)\right) \\ &\quad - \frac{M^{2_\mu^*}t^{2_\mu^*}}{2_\mu^*\lambda^{1-\beta/\alpha}}. \end{aligned}$$

It is clear that $\lim\limits_{t\to\infty} g(t) = -\infty$, therefore, the function $g(t)$ possesses a maximum value at the point,

$$t_\varepsilon = \left(\frac{\lambda^{1-\beta/\alpha}}{M^{2_\mu^*-2}}\left(\frac{1}{\lambda^{1-\beta/\alpha}}[\kappa_\mu S(\mu,N) + O(\varepsilon^{N-2\mu})] + \frac{\rho^2}{\lambda^{\beta/\alpha}}[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] - 2F(\varepsilon)\right)\right)^{\frac{1}{2_\mu^*-2}}.$$

Moreover, at this point $t_{\lambda,\varepsilon}$ we have,

$$\begin{aligned} h(t_{\lambda,\varepsilon}) = &\left(\left(\lambda^{1-\beta/\alpha}\right)^{\frac{2}{2_\mu^*}}\left(\frac{1}{\lambda^{1-\beta/\alpha}}[\kappa_\mu S(\mu,N) + O(\varepsilon^{N-2\mu})] + \frac{\rho^2}{\lambda^{\beta/\alpha}}[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] - 2F(\varepsilon)\right)\right)^{\frac{2_\mu^*}{2_\mu^*-2}} \\ &\times\left(\frac{1}{2} - \frac{1}{2_\mu^*}\right) \end{aligned}$$

To complete the proof we must show that the inequality

$$h(t_{\lambda,\varepsilon}) < c_\mu^* := \frac{1}{\lambda^{1-\beta/\alpha}}\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)(\kappa_\mu S(\mu,N))^{\frac{2_\mu^*}{2_\mu^*-2}}, \tag{5.3.27}$$

holds true for ε small enough. Thus, simplifying (5.3.27), we are left to choose $\rho > 0$ such that inequality

$$O(\varepsilon^{N-2\mu}) + \rho^2[\kappa_\beta S(\beta,N) + O(\varepsilon^{N-2\beta})] < 2\lambda^{\beta/\alpha}F(\varepsilon).$$

holds true provided ε is small enough. To this end, take $\rho = \varepsilon^\delta$ with $\delta > 0$ to be determined, therefore, since

$$O(\varepsilon^{N-2\mu}) + \kappa_\beta S(\beta,N)\varepsilon^{2\delta} + O(\varepsilon^{N-2\beta+2\delta}) = O(\varepsilon^\tau),$$

with $\tau = \min\{N-2\mu, 2\delta, N-2\beta+2\delta\} = \min\{N-2\mu, 2\delta\}$, the proof will be completed once we choose $\delta > 0$ such that the inequality

$$O(\varepsilon^\tau) < 2\lambda^{\beta/\alpha}F(\varepsilon), \tag{5.3.28}$$

holds true for ε small enough. If $N = 2\alpha$, because of (5.3.25), inequality (5.3.28) reads,

$$O(\varepsilon^\tau) < 2\lambda^{\beta/\alpha}\varepsilon^{\alpha+\delta}|\log(\varepsilon)|. \tag{5.3.29}$$

Since $\varepsilon << 1$, inequality (5.3.29) holds if and only if $\tau = \min\{2\alpha-2\mu, 2\delta\} = \min\{2\beta, 2\delta\} > \alpha+\delta$. Using the identity $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$, we find that $\tau > \alpha+\delta$ implies $\beta+\delta-|\beta-\delta| > \alpha+\delta$, which is impossible because $\alpha > \beta$. Therefore, (5.3.28) can not hold if $N = 2\alpha$. On the other hand, if $N > 2\alpha$, inequality (5.3.28) has the form,

$$O(\varepsilon^\tau) < 2\lambda^{\beta/\alpha}\varepsilon^{\alpha+\delta}. \tag{5.3.30}$$

Since $\varepsilon << 1$, inequality (5.3.30) holds if and only if $\tau = \min\{N-2\mu, 2\delta\} > \alpha+\delta$. Keeping in mind the identity $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$, if $\tau > \alpha+\delta$ we arrive at the condition

$$N-2\mu-|N-2\mu-2\delta| > 2\alpha. \tag{5.3.31}$$

Consequently, we have two options:

(1) $N-2\mu > 2\delta$ combined with (5.3.31) provides us with the range,

$$N-2\mu > 2\delta > 2\alpha. \tag{5.3.32}$$

Then $N > 4\alpha-2\beta$ necessarily, so that we can choose a positive δ satisfying (5.3.32) and, hence, inequality (5.3.28) holds for ε small enough.

(2) $N-2\mu < 2\delta$ combined with (5.3.31) implies that $2(N-2\mu)-2\alpha > 2\delta$, and hence,

$$2(N-2\mu)-2\alpha > 2\delta > N-2\mu. \tag{5.3.33}$$

Once again $N > 4\alpha-2\beta$ necessarily, so that we can choose a positive δ satisfying (5.3.33) and, hence, inequality (5.3.28) holds for ε small enough.

□

We finish this chapter by completing the proof of Theorem 5.1.2, dealing with the remaining case $\alpha \neq 2\beta$.

PROOF OF THEOREM 5.1.2. CASE $\alpha \neq 2\beta$. Due to Lemma 5.3.6, we obtain,

$$c_\varepsilon \leq \sup_{t\geq 0} \mathcal{H}_\lambda^{\alpha,\beta}(t\overline{w}_\varepsilon^\mu, t\overline{z}_\varepsilon^\beta) < c_\mu^*,$$

for ε small enough and thanks to Lemma 5.3.3 the functional $\mathcal{H}_\lambda^{\alpha,\beta}$ satisfies the PS condition for any level c_ε with ε small enough. Moreover, by Lemma 5.2.3, the functional $\mathcal{H}_\lambda^{\alpha,\beta}$ has the mountain pass geometry. Therefore, we can apply the Mountain Pass Theorem to obtain the existence of a critical point $(w,z) \in \mathcal{X}_0^\mu(\mathscr{C}_\Omega) \times \mathcal{X}_0^\beta(\mathscr{C}_\Omega)$. The rest follows as in the subcritical case. □

REMARK 5.3.2. *As final comment, we observe the following facts about the behaviour of problem* $(P_\lambda^{\alpha,\beta})$ *for some particular choices of parameters* α *and* β.

- *Since* $0 < \beta < 1$ *and* $\beta < \alpha < 1+\beta$ *it follows that* $4\alpha > \alpha - 2\beta$. *Taking* $\beta \to 0$ *in Theorem 5.1.2 we recover* [**18**, Theorem 1.2].
- *Fixed* $\beta \in (0,1)$ *and letting* $\mu := \alpha - \beta \to 1$, *i.e.* $\alpha \to 1+\beta$, *problem* $(P_\lambda^{\alpha,\beta})$ *reads*

$$(-\Delta)u = \lambda(-\Delta)^{-\beta}u + |u|^{p-1}u, \quad u = 0, \quad \text{on } \partial\Omega. \tag{5.3.34}$$

Assuming that $\partial\Omega$ *is a* $\mathcal{C}^2$ *manifold, by standard elliptic regularity theory,* [**47**, Sec. 8.3, Theorem 1], *it follows that* $u \in H_0^1(\Omega) \cap H^2(\Omega)$. *Thus, we can apply the operator* $(-\Delta)^\beta$ *to equation* (5.3.34) *and, as a consequence of Theorems 5.1.1 and 5.1.2, we obtain the existence of a positive solution to the problem*

$$\begin{cases} (-\Delta)^{1+\beta}u = \lambda u + (-\Delta)^\beta |u|^{p-1}u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

CHAPTER 6

Homotopy Regularization for a High-Order parabolic equation

In this final chapter we study the Cauchy Problem for a quasilinear degenerate high-order parabolic equation

$$\begin{cases} u_t = (-1)^{m-1}\nabla\cdot(f^n(|u|)\nabla\Delta^{m-1}u) & \text{in } \mathbb{R}^N\times\mathbb{R}_+, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

with $m\in\mathbb{N},\ m>1$ and $n>0$ a fixed exponent. Moreover, we assume that f is a continuous monotone increasing positive bounded function with $f(0)=0$ and the initial data $u_0(x)$ is bounded smooth and compactly supported. Thus, through an homotopy argument based on an analytic ε-regularization of the degenerate term $f^n(|u|)$ we are able to extract information about the solutions inherited from the polyharmonic equation when $n=0$.

6.1. Introduction

We conclude this PhD thesis dissertation with the study of the well-posedness of the Cauchy Problem for a quasilinear degenerate high-order parabolic equation of the form

$$(P_{HG})\qquad \begin{cases} u_t = (-1)^{m-1}\nabla\cdot(f^n(|u|)\nabla\Delta^{m-1}u) & \text{in } \mathbb{R}^N\times\mathbb{R}_+, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

with $m\in\mathbb{N},\ m>1$ and $n>0$ is a fixed exponent, f is a continuous monotone increasing positive bounded function with $f(0)=0$ and the initial data $u_0(x)$ is a bounded smooth compactly supported function.

The principal issue to overcome in this chapter is to detect proper solutions to the Cauchy Problem for the degenerate equation (P_{HG}) by uniformly parabolic analytic ε-regularizations. To this end, following the work [**8**], we use an analytic homotopy approach based on *a priori* estimates for solutions to uniformly parabolic analytic ε-regularization equations, namely

$$(6.1.1)\qquad \begin{cases} u_t = (-1)^{m-1}\nabla\cdot(\phi_\varepsilon(u)\nabla\Delta^{m-1}u) & \text{in } \mathbb{R}^N\times\mathbb{R}_+, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where $\phi_\varepsilon(u)$, $\varepsilon\in(0,1]$ is an analytic ε-regularization such that $\phi_0(u)=f^n(|u|)$ and $\phi_1(u)=1$ using a classic technique relying on integral identities for weak solutions.

Next, we study an analytic homotopy transformation in both parameters, $\varepsilon\to 0^+$ and $n\to 0^+$ and describe branching of solutions to (P_{HG}) from the polyharmonic heat equation

$$(6.1.2)\qquad \begin{cases} u_t = -(-\Delta)^m u & \text{in } \mathbb{R}^N\times\mathbb{R}_+, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

which provides some qualitative oscillatory properties (changing sign) as well as the uniqueness of solutions to (P_{HG}), at least for small $n>0$. Thus, the homotopic deformation is a continuous deformation from solutions to (P_{HG}) to solutions to (6.1.2) for which important information is inherited. The case $m=2$ and $f(t)=t$ has been studied in [**8**], however,

in this chapter we generalize the degenerate term $f^n(|u|)$ and under some assumptions we are able to perform an homotopy argument which provides us with the unique solutions to (P_{HG}) at least when the parameter n is very close to zero.
Now, we introduce the homotopy technique. We say that (P_{HG}) is homotopic to the ε-regularized equation (6.1.1) if there exists a family of uniformly parabolic equations (the homotopic deformation) with a coefficient,

$$\phi_\varepsilon(u) > 0, \quad \text{analytic in both variables } u \in \mathbb{R},\ \varepsilon \in (0,1],$$

with unique analytic solutions $u_\varepsilon(x,t)$ of the problem

$$\begin{cases} u_t = (-1)^{m-1}\nabla \cdot (\phi_\varepsilon(u)\nabla\Delta^{m-1}u) & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

such that $\phi_1(u) = 1$ and $\phi_\varepsilon(u) \to f^n(|u|)$ uniformly on compact sets as $\varepsilon \to 0^+$. Based on the ideas of [**8**] we choose the homotopic path to be

$$\phi_\varepsilon(u) = f^n(\varepsilon) + (1-\varepsilon) f^n\left((\varepsilon^2 + u^2)^{1/2}\right). \tag{6.1.3}$$

Moreover, using classic parabolic theory (see for instance [**46, 50**]) the non-degenerate equation (6.1.1) has a unique classical solution $u_\varepsilon(x,t)$ analytic in the variables ε, x, t.

Therefore, we can know define what a proper solution is in the following terms.

DEFINITION 6.1.1. *We say that $u(x,t)$ is a proper solution to the Cauchy Problem (P_{HG}) if*

$$u_\varepsilon(x,t) \to u(x,t), \ \text{ as } \varepsilon \to 0^+, \tag{6.1.4}$$

where $\{u_\varepsilon(x,t)\}_{\varepsilon\in(0,1]}$ is the family of classical global solutions to the regularized Cauchy Problem (6.1.1)

As we will see, due to the similarity of the expressions for weak solutions to the Cauchy Problem (P_{HG}) and the Free Boundary Problem corresponding to the evolution of the support of the solution of (P_{HG}) the previous analysis based on ε-regularizations is unable to distinguish both type of solutions. Another issue that arises when applying these ε-regularizations is the uniqueness of the limit of $u_\varepsilon(x,t)$ as $\varepsilon \to 0^+$. In the case $f(t) = t$, thanks to the scaling properties of $f(t)$, this problem is studied with an affirmative conclusion; see [**8**]. However, due to the nature of the term $f(|u|)$, we will be unable to provide a similar result for problem (P_{HG}). Also, we can not discard the dependence of the solution from the type of analytic ε-regularization $\phi_\varepsilon(u)$. Hence, we must carry out alternative arguments which could solve some of the issues explained above. Subsequently, after this limit procedure in the ε-regularization we perform a second limit as $n \to 0^+$, i.e., a continuous connection with solutions to the polyharmonic heat equation (6.1.2),

$$u(x,t) \to u_{PH}(x,t), \quad \text{as } n \to 0^+.$$

Finally, we perform a double limit $n, \varepsilon \to 0^+$ from which we obtain the conditions on the parameters ε and n needed to obtain such a functional convergence. As we have said, performing that limit over integral identities defining weak solutions results inconclusive to determine proper solutions to the Cauchy Problem from those to the FBP. To carry out this step we choose the simpler path

$$\phi_\varepsilon(u) = f^n\left((\varepsilon^2 + u^2)^{1/2}\right).$$

Now we state the main result of the chapter.

THEOREM 6.1.1. *Suppose that*

$$n|\ln f(\varepsilon(n))| \to 0, \quad \text{as } n \to 0^+,$$

and the regularization family $\{u_\varepsilon(x,t)\}_{\varepsilon\in(0,1]}$ *is uniformly bounded. Then*

(1) The solution $u(x,t)$ *to the regularized problem*

$$\begin{cases} u_t = (-1)^{m-1}\nabla\cdot\left(f^n\left((\varepsilon^2+u^2)^{1/2}\right)\nabla\Delta^{m-1}u\right) & \text{in } \mathbb{R}^N\times\mathbb{R}_+, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

converges uniformly to the solution $u_{PH}(x,t)$ *to the polyharmonic heat equation* (6.2.1) *as* $n\to 0^+$ *and* $\varepsilon \to 0^+$.

(2) If the convolution

$$\varphi(x,t) = -\int_0^t \nabla H(x,t-s) * \ln|u_{PH}(x,s)|\nabla\Delta^{m-1}u_{PH}(x,s)ds,$$

remains bounded for the solution to the polyharmonic heat equation (6.2.1)*, the rate of convergence as* $n \to 0^+$ *of the asymptotic expansion* $u(x,t) = u_{PH}(x,t) + V$ *is given by*

$$V := n\varphi + o(n).$$

Thanks to the previous theorem we can assert that there exists a branch of solutions to the high-order equation (P_{HG}) emanating at $n^+ = 0$ from the unique solution of the polyharmonic heat equation (6.2.1).
Finally, let us observe that problem (P_{HG}) is written for solutions of changing sign, which can occur in the Cauchy Problem, because of the branching from the polyharmonic heat equation (6.1.2), and also in some free-boundary problems (See [**8**]) related. Our main convention, according to [**8**], is that oscillatory sign changing solutions are related to the Cauchy Problem while non-negative solutions are left for the Free Boundary Problem corresponding to the evolution of the support of the solution of (P_{HG}). On the other hand, for both problems, the Cauchy problem and the FBP, it is commonly assumed that the solutions satisfy the following standard free boundary conditions:

$$\begin{cases} u=0 & \text{zero-height,} \\ \nabla u = \Delta u = \nabla\Delta u = \ldots = \Delta^{m-1} = 0 & \text{zero contact angle,} \\ \overline{\mathbf{n}}\cdot\left(f^n(|u|)\nabla\Delta^{m-1}u\right) = 0 & \text{zero-flux} \end{cases} \tag{6.1.5}$$

at the interface $\Gamma_0[u]$, i.e., the lateral boundary

$$supp\ u \subset \mathbb{R}^N \times \mathbb{R}_+.$$

Let us also note that, under the zero-flux condition. the total mass,

$$M(x,t) := \int u(x,t)dx$$

is preserved, since differentiating under the integral sign with respect to the temporal variable and using the Divergence Theorem,

$$\frac{d}{dt}M(x,t) = (-1)^{m-1}\int_{\Gamma_0\cap\{t\}} \overline{\mathbf{n}}\cdot\left(f^n(|u|)\nabla\Delta^{m-1}u\right)d\sigma = 0.$$

Because of those assumptions, the expressions for solutions for the Cauchy Problem (P_{HG}) and the FBP problem coincide. As our analysis is mainly based on those integral identities, it will be unable to difference both solutions.

6.2. Polyharmonic heat equation when $n = 0$

To study the well-posedness of the Cauchy Problem (P_{HG}) we use an analytic homotopic deformation from (P_{HG}) to an equation that provides us with some useful information of its solutions, namely, to the polyharmonic heat equation,

$$\begin{cases} u_t = -(-\Delta)^m u & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{6.2.1}$$

This equation has been extensively studied in the last years [**17, 58, 52**]. It is well know that for smooth compactly supported initial data $u_0(x)$, satisfying a growth condition at infinity, see [**46**],

$$u_0(x) \in L^2_\rho(\mathbb{R}^N), \quad \rho(x) = e^{a|x|^\alpha}, \tag{6.2.2}$$

for some constant $a > 0$ and $\alpha = \frac{2m}{2m-1}$, the polyharmonic heat equation (6.2.1) admits an unique classic solution given by the Poisson-type integral,

$$u_{PH}(x,t) = \mathcal{H}(x,t) * u_0(x) = t^{-\frac{N}{2m}} \int_{\mathbb{R}^N} F\left((x-z)t^{-\frac{1}{2m}}\right) u_0(z)dz,$$

where $\mathcal{H}(x,t)$ is the fundamental solution for (6.2.1),

$$\mathcal{H}(x,t) = t^{-\frac{N}{2m}} F\left(\frac{x}{t^{\frac{1}{2m}}}\right),$$

such that the rescaled kernel $F(y)$, with $y = \frac{x}{t^{\frac{1}{2m}}}$, is the unique radial solution of the elliptic equation

$$\mathcal{L}[F] \equiv -(-\Delta)^m F + \frac{1}{2m} y \cdot \nabla F + \frac{N}{2m} F = 0, \quad \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} F(y)dy = 1. \tag{6.2.3}$$

It can be seen, [**46**], that the profile function $F(y)$ decays exponentially at infinity. Specifically, there exists some positive constants $C > 1$, $a > 0$ depending on N and m such that

$$|F(y)| \leq C\omega e^{-a|y|^\alpha}, \quad \text{in } \mathbb{R}^N, \ \alpha = \frac{2m}{2m-1} \text{ and } \omega = \int_{\mathbb{R}^N} e^{-a|y|^\alpha} dy.$$

On the other hand, using the Fourier Transform (see for instance [**30, 46**]) the profile $F(y)$ is also given by the expression

$$F(y) = F_{m,N}(y) = |y|^{1-N} \int_0^\infty e^{-s^{2m}} (|y|s)^{\frac{N}{2}} J_{\frac{N-2}{2}}(|y|s)ds, \tag{6.2.4}$$

where J_k is the k-th Bessel function of first kind. Note that thanks to (6.2.4) and contrary to what happens in the case $m = 1$ where the profile function is the well known Gaussian function, we know that the kernel $F_{m,N}(y)$ depends not only on the parameter m but also on the dimension N.

Moreover, due to the presence of the Bessel functions in the integral expression of $F(y)$, the solutions to the polyharmonic heat equation are oscillatory functions. Another big difference between the case $m = 1$ and $m > 1$. While in the first case the positivity of the solutions is preserved, this is no longer true for solutions to (6.2.1) with $m > 1$. Nevertheless, those solutions exhibits what is called (see for instance [**49, 56**]) *eventual positivity*, i.e. there exists a time $T = T(u_0(x), K) > 0$ such that for any compact set $K \subset \mathbb{R}^N$ and any compactly supported initial data $u_0(x)$,

$$u_{PH}(x,t) > 0, \ \forall x \in K, \ \forall t > T.$$

To finish this brief exposition for some of the properties of the polyharmonic equation (6.2.1), let us recall some facts about the spectrum of the operator $\mathcal{L}$ denoted by (6.2.3). As it is easily verified, for $m > 1$ the operator $\mathcal{L}$ is not symmetric and does not admit a self-adjoint extension. Ascribing to the operator $\mathcal{L}$ the domain $H^{2m}_\rho(\mathbb{R}^N)$ it can be proved, see [**45, 53**], the following.

Lemma 6.2.1.

- *The operator $\mathcal{L} : H^{2m}_\rho(\mathbb{R}^N) \mapsto L^2_\rho(\mathbb{R}^N)$ is a bounded operator with only the real point spectrum*

$$\sigma(\mathcal{L}) = \left\{ \lambda_\beta = -\frac{|\beta|}{2m}, |\beta| = 0, 1, 2, \dots \right\}.$$

  Eigenvalues λ_β have finite multiplicity with eigenfunctions

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \equiv \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \left(\frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial y_N}\right)^{\beta_N} F(y).$$

- *The set of eigenfunctions $\Phi = \{\psi_\beta, |\beta| = 0, 1, 2, \dots\}$ is complete in $L^2_\rho(\mathbb{R}^N)$*

In the classical case $m = 1$, where the profile $F(y)$ is the rescaled Gaussian kernel, the eigenfunctions $\psi_\beta(y)$ are given by

$$\psi_\beta(y) = e^{-\frac{|y|^2}{4}} \mathcal{H}_\beta(y), \quad \mathcal{H}_\beta(y) \equiv \mathcal{H}_{\beta_1}(y_1) \dots \mathcal{H}_{\beta_N}(y_N),$$

where $\mathcal{H}_\beta$ denote the Hermite polynomials in $\mathbb{R}^N$. The operator $\mathcal{L}$ with the domain $H^2_\rho(\mathbb{R}^N)$, $\rho = e^{\frac{|y|^2}{4}}$, is self-adjoint and the eigenfunctions form an orthonormal basis in $L^2_\rho(\mathbb{R}^N)$. In [**45**] it is also proved that the adjoint operator,

$$\mathcal{L}^* = -(-\Delta)^m - \frac{1}{2m} y \cdot \nabla,$$

possesses a set of eigenfunctions that forms an orthonormal basis in $L^2_{\rho^*}(\mathbb{R}^N)$, with the specific exponentially decaying weight function $\rho^*(y) = e^{-a|y|^\alpha}$. Moreover, the operator $\mathcal{L}^* : H^{2m}_{\rho^*}(\mathbb{R}^N) \mapsto L^2_{\rho^*}(\mathbb{R}^N)$ is a bounded linear operator,

$$\langle \mathcal{L}[v], w \rangle = \langle v, \mathcal{L}^*[w] \rangle \quad \text{for any } v \in H^{2m}_\rho(\mathbb{R}^N), \ w \in H^{2m}_{\rho^*}(\mathbb{R}^N),$$

and $\sigma(\mathcal{L}^*) = \sigma(\mathcal{L})$ with the eigenfunctions $\{\psi^*_\beta(y)\}$ being polynomials of order $|\beta|$,

$$\sqrt{\beta!}\psi^*_\beta(y) = y^\beta + \sum_{j=1}^{\left[\frac{|\beta|}{2m}\right]} \frac{1}{j!} (-\Delta)^{mj} y^\beta.$$

6.3. Preliminary estimates: Bernis–Friedman-type inequalities

Throughout this section, for any $\varepsilon \in (0,1]$ let $u_\varepsilon(x,t)$ be the solution of Cauchy Problem for the regularized non-degenerate uniformly parabolic equation (6.1.1). By classic parabolic theory [**46, 50**] this family is continuous and analytic in $\varepsilon \in (0,1]$ in the appropriate functional topology, at least in some interval $[0,T]$. Moreover, all the derivatives are Hölder continuous in $\overline{\Omega} \times [0,T]$. From now on, we denote with Ω either $\mathbb{R}^N$ or, equivalently, the bounded domain *supp* $u \cap \{t\}$ (the section of the support).

The following result comes from similar ideas as those performed by Bernis-Friedman [**21**] and will be used in the sequel to prove some of the main results of this chapter.

PROPOSITION 6.3.1. *Let $u_\varepsilon(x,t)$ be the unique global solution to the Cauchy Problem for the regularized non-degenerate equation* (6.1.1). *Then for $t \in [0,T]$, there exists $K > 0$ independent of ε and T such that for $j \in \mathbb{N}$,*

(1) $\displaystyle\int_\Omega |\Delta^{\frac{m-1}{2}} u_\varepsilon(x,t)|^2 dx \le K$ *if* $m = 2j+1$.
(2) $\displaystyle\int_\Omega |\nabla\Delta^{\frac{m-2}{2}} u_\varepsilon(x,t)|^2 dx \le K$ *if* $m = 2j$.
(3) $\displaystyle\int_\Omega |\Delta^{\frac{m-2}{2}} u_\varepsilon(x,t)|^2 dx \le K$, *if* $m = 2j$, $j \in \mathbb{N}$.
(4) $\displaystyle\int_\Omega u_\varepsilon(x,t) dx \le K$.
(5) *Setting* $h_\varepsilon = \phi_\varepsilon(u_\varepsilon)\nabla\Delta^{m-1}u_\varepsilon$, *we have* $||h_\varepsilon||_{L^2(\Omega\times(0,t))} \le K$.

PROOF. First we note that, thanks to the boundary conditions (6.1.5),

$$-\int_\Omega u_\varepsilon(x,\cdot)\Delta^{m-1}u_\varepsilon(x,\cdot)dx = \begin{cases} (-1)^m \displaystyle\int_\Omega |\Delta^{\frac{m-1}{2}} u_\varepsilon(x,\cdot)|^2 dx & \text{if } m = 2j+1, \\[2ex] (-1)^m \displaystyle\int_\Omega |\nabla\Delta^{\frac{m-2}{2}} u_\varepsilon(x,\cdot)|^2 dx & \text{if } m = 2j, \end{cases}$$

for $j \in \mathbb{N}$, as well as

$$\int_\Omega u_\varepsilon(x,t+h)\Delta^{m-1}u_\varepsilon(x,t)dx = \int_\Omega u_\varepsilon(x,t)\Delta^{m-1}u_\varepsilon(x,t+h)dx.$$

Hence,

$$-\int_\Omega [\Delta^{m-1}u_\varepsilon(x,t+h) + \Delta^{m-1}u_\varepsilon(x,t)][u_\varepsilon(x,t+h) - u_\varepsilon(x,t)]dx =$$
$$= \begin{cases} (-1)^m \displaystyle\int_\Omega |\Delta^{\frac{m-1}{2}} u_\varepsilon(x,t+h)|^2 - |\Delta^{\frac{m-1}{2}} u_\varepsilon(x,t)|^2 dx & \text{if } m = 2j+1, \\[2ex] (-1)^m \displaystyle\int_\Omega |\nabla\Delta^{\frac{m-2}{2}} u_\varepsilon(x,t+h)|^2 - |\nabla\Delta^{\frac{m-2}{2}} u_\varepsilon(x,t)|^2 dx & \text{if } m = 2j. \end{cases}$$

Then, dividing by h, taking the limit as $h \to 0^+$ and integrating between 0 and $t \in [0,T]$ we get

$$
\begin{aligned}
&-\iint\limits_{\Omega\times(0,t)} \Delta^{m-1}u_\varepsilon(x,t)u_{\varepsilon,t}(x,t)\,dx\,dt \\
&= \begin{cases} \dfrac{(-1)^m}{2}\displaystyle\int_\Omega |\Delta^{\frac{m-1}{2}}u_\varepsilon(x,t)|^2 - |\Delta^{\frac{m-1}{2}}u_\varepsilon(x,0)|^2dx & \text{if } m=2j+1, \\[2ex] \dfrac{(-1)^m}{2}\displaystyle\int_\Omega |\nabla\Delta^{\frac{m-2}{2}}u_\varepsilon(x,t)|^2 - |\nabla\Delta^{\frac{m-2}{2}}u_\varepsilon(x,0)|^2dx & \text{if } m=2j. \end{cases}
\end{aligned}
\tag{6.3.1}
$$

Now, multiplying the regularized equation (6.1.1) by $\Delta^{m-1}u_\varepsilon$, integrating by parts in $\Omega\times(0,t)$ and using the boundary conditions, we obtain

$$
\iint\limits_{\Omega\times(0,t)} \nabla\cdot(\phi_\varepsilon(u_\varepsilon)\nabla\Delta^{m-1}u_\varepsilon)\Delta^{m-1}u_\varepsilon\,dx\,dt = \iint\limits_{\Omega\times(0,t)} \phi_\varepsilon(u_\varepsilon)|\nabla\Delta^{m-1}u_\varepsilon|^2\,dx\,dt. \tag{6.3.2}
$$

Therefore, from (6.3.1) and (6.3.2), we conclude

$$
\int_\Omega |\Delta^{\frac{m-1}{2}}u_\varepsilon(x,0)|^2dx = \int_\Omega |\Delta^{\frac{m-1}{2}}u_\varepsilon(x,t)|^2\,dx\,dt + 2\iint\limits_{\Omega\times(0,t)} \phi_\varepsilon(u_\varepsilon)|\nabla\Delta^{m-1}u_\varepsilon|^2\,dx\,dt
$$

if $m=2j+1$, and

$$
\int_\Omega |\nabla\Delta^{\frac{m-2}{2}}u_\varepsilon(x,0)|^2dx = \int_\Omega |\nabla\Delta^{\frac{m-2}{2}}u_\varepsilon(x,t)|^2\,dx\,dt + 2\iint\limits_{\Omega\times(0,t)} \phi_\varepsilon(u_\varepsilon)|\nabla\Delta^{m-1}u_\varepsilon|^2\,dx\,dt,
$$

if $m=2j$. Consequently, due to these Bernis-Friedman-type inequalities we have proved assertions (1) and (2). Let us observe that from the above integral equalities we also get

$$
\iint\limits_{\Omega\times(0,t)} \phi_\varepsilon(u_\varepsilon)|\nabla\Delta^{m-1}u_\varepsilon|^2\,dx\,dt \le K,
$$

and, therefore,

$$
f^n(\varepsilon)\iint\limits_{\Omega\times(0,t)} |\nabla\Delta^{m-1}u_\varepsilon|^2\,dx\,dt \le K, \tag{6.3.3}
$$

$$
\iint\limits_{\Omega\times(0,t)} f^n\left((\varepsilon^2+u_\varepsilon^2)^{1/2}\right)|\nabla\Delta^{m-1}u_\varepsilon|^2\,dx\,dt \le K, \tag{6.3.4}
$$

with K as a positive constant. For unbounded domains such as $\Omega=\mathbb{R}^N$ the results are true thanks to the exponential decay of solutions (so that the integration by parts is justified). Thus, the inequalities remain true in certain $L^2_\rho(\mathbb{R}^N)$ and $H^{2m}_\rho(\mathbb{R}^N)$ weighted spaces for an appropriate weight. Moreover, from the conservation of mass and the boundary conditions (6.1.5), it also follows that

$$
\int_\Omega u_\varepsilon(x,t)dx \le K,\ \forall t\in[0,T].
$$

On the other hand, applying Poincaré's inequality in the case $m = 2j$ (assuming a bounded domain Ω for the FBP) we find

$$\int_\Omega |\Delta^{\frac{m-2}{2}} u(x,t)|^2 \, dx \, dt \leq K,$$

and we conclude (3). Finally, we prove (5). Since f is a bounded function, i.e. $\sup_{t\in\mathbb{R}_+} f(t) \leq C_f$, it follows that

$$\begin{aligned}
\iint\limits_{\Omega\times(0,t)} |h_\varepsilon|^2 dx dt &= \iint\limits_{\Omega\times(0,t)} \phi_\varepsilon^2(u_\varepsilon)|\nabla\Delta^{m-1}u_\varepsilon|^2 \, dx \, dt \\
&= \iint\limits_{\Omega\times(0,t)} \left(f^n(\varepsilon) + (1-\varepsilon) f^n\left((\varepsilon^2 + u_\varepsilon^2)^{1/2}\right)\right)^2 |\nabla\Delta^{m-1}u_\varepsilon|^2 \, dx \, dt \\
&\leq 2 f^{2n}(\varepsilon) \iint\limits_{\Omega\times(0,t)} |\nabla\Delta^{m-1}u_\varepsilon|^2 \, dx \, dt \\
&+ 2C_f^n \iint\limits_{\Omega\times(0,t)} f^n\left((\varepsilon^2 + u_\varepsilon)^{1/2}\right) |\nabla\Delta^{m-1}u_\varepsilon|^2 \, dx \, dt \\
&\leq 2KC_f^n.
\end{aligned}$$

□

Additionally, we obtain uniform L^∞ estimates for solutions to (P_{HG}) by means of a scaling technique, [**60**].

PROPOSITION 6.3.2. *Any solution to problem (P_{HG}) is uniformly bounded.*

PROOF. We argue by contradiction. Assume that there exists a monotone sequence $\{t_k\} \to T$ and $\{x_k\} \subset \mathbb{R}^N$ such that

$$\sup_{(x,t)\in\mathbb{R}^N\times(0,t_k)} |u(x,t)| = |u(x_k,t_k)| = C_k \to +\infty \quad \text{monotonically.} \tag{6.3.5}$$

Subsequently, we rescale the solution $u(x,t)$ to (P_{HG}) and define the sequence $\{v_k(y,s)\}$ as follows,

$$v_k(y,s) := \frac{1}{C_k} u\left(\lambda_k y + x_k, \lambda_k^{2m} s + t_k\right),$$

for some positive number λ_k (to be specified later) such that $\{\lambda_k\} \to 0$. Thus, with this rescaling we just perform a zoom around the point (x_k, t_k) in the region $B_{\delta/\lambda_k}(0) \times \left(\frac{-t_k}{\lambda_k^{2m}}, 0\right)$, for $\delta > 0$ sufficiently small and where $B_{\delta/\lambda_k}(0)$ is the ball of radius $\frac{\delta}{\lambda_k}$ and centered at the origin. Therefore, due to the scaling and assumption (6.3.5) it is now clear that,

$$|v_k(0,0)| = 1 \quad \text{and} \quad |v_k(y,s)| \leq 1, \quad \text{for all } k \geq 1 \text{ and } s \in \left[-\frac{t_k}{\lambda_k^{2m}}, 0\right). \tag{6.3.6}$$

Moreover, the function v_k satisfies the equation

$$\frac{\partial}{\partial s} v_k = (-1)^{m-1} \nabla \cdot (f^n(|C_k v_k|) \nabla\Delta^{m-1} v_k), \tag{6.3.7}$$

for any $(y,s)\in\mathbb{R}^N\times(-\frac{t_k}{\lambda_k^{2m}},0)$ with initial data $v_{k0}(y)=\dfrac{1}{C_k}u_0\left(\lambda_k y+x_k\right)$. On the other hand, thanks to the uniform estimate (1) in Proposition 6.3.1, for a positive constant K, we obtain

$$\int_\Omega |\Delta^{\frac{m-1}{2}}u(x,t)|^2dx=\frac{C_k^2}{\lambda_k^{N+2(m-1)}}\int_{\Omega_k}|\Delta^{\frac{m-1}{2}}v_k(y,s)|^2dy\le K,$$

so that

$$\int_{\Omega_k}|\Delta^{\frac{m-1}{2}}v_k(y,s)|^2dy\le\frac{\lambda_k^{N+2(m-1)}}{C_k^2}K,$$

if $m=2j+1$. In a similar way, if $m=2j$, from (2) in Proposition 6.3.1 we find,

$$\int_{\Omega_k}|\nabla\Delta^{\frac{m-2}{2}}v_k(y,s)|^2dy\le\frac{\lambda_k^{N+2(m-1)}}{C_k^2}K.$$

Moreover, using (3) in Proposition 6.3.1,

$$\int_{\Omega_k}|\Delta^{\frac{m-2}{2}}v_k(y,s)|^2dy\le\frac{\lambda_k^{N+2(m-1)}}{C_k^2}K_1.$$

Hence, passing to the limit as $k\to\infty$, along a subsequence if necessary, the limit function $v_k\to v(y,s)$ satisfies,

$$\int_{\mathbb{R}^N}|\Delta^{\frac{m-1}{2}}v(y,s)|^2dy=0\quad\text{if } m=2j+1,\tag{6.3.8}$$

and

$$\int_{\mathbb{R}^N}|\Delta^{\frac{m-2}{2}}v(y,s)|^2dy=0\quad\text{if } m=2j.\tag{6.3.9}$$

Therefore, passing to the limit and using (6.3.8) and (6.3.9) together with the boundary conditions (6.1.5), we find that the limit function satisfies

$$\begin{cases}\Delta^{\widetilde{m}}v=0 & \text{in } \mathbb{R}^N,\\ |v|\le 1, & \\ \lim\limits_{|y|\to\infty}v(y,\cdot)=0. &\end{cases}$$

with $\widetilde{m}=\frac{m-1}{2}$ if $m=2j+1$ and $\widetilde{m}=\frac{m-2}{2}$. Therefore, because of a Liouville-type Theorem, see [**13, 64**], we obtain that v has to be constant, and due to the condition at infinity we conclude that $v\equiv 0$ in contradiction with (6.3.6). Consequently, we conclude, from the construction of the functions v_k and the limiting argument performed above, that $v\equiv 0$ in contradiction with (6.3.6). Actually, (6.3.6) implies, by interior parabolic regularity, that $v(y,0)$ must be non-trivial in a neighbourhood of $y=0$. $\square$

6.4. Homotopy deformations

Next we show the existence of solutions to the Cauchy Problem (P_{HG}) using a limiting argument as

- $\varepsilon\to 0^+$, obtaining the convergence of solutions to the regularized problem (6.1.1) to solutions to problem (P_{HG}).

- $\varepsilon \to 0^+$ and $n = n(\varepsilon) \to 0^+$, obtaining the convergence of solutions to problem (P_{HG}) to solutions to the polyharmonic heat equation (6.2.1) under some conditions on the behavior of $n(\varepsilon)$ for $\varepsilon \approx 0$.

As the former procedure is unable to distinguish proper solutions to the Cauchy problem (P_{HG}) from solutions to the FBP we perform a second homotopic argument as

- $n \to 0^+$ and $\varepsilon = \varepsilon(n) \to 0^+$ as $n \to 0^+$.

First we recall the following Lemma due to Aubin and Lions, see [**14**].

LEMMA 6.4.1. *Let $X_0 \subseteq X \subset X_1$ be three Banach spaces such that X_0 is compactly embedded in X and X is continuously embedded in X_1. For $1 \le p, q \le \infty$, let*

$$W = \{u \in L^p([0,T], X_0),\ u_t \in L^q([0,T], X_1)\}.$$

- *If $p < \infty$ then the embedding of W into $L^p([0,T], X)$ is compact.*
- *If $p = \infty$ and $q > 1$ then the embedding of W into $C([0,T], X)$ is compact.*

First, for bounded domains Ω and due to Proposition 6.3.1 together with Lemma 6.4.1 we can extract a convergent subsequence in $L^2(\Omega \times [0,T])$ as $\varepsilon \to 0^+$ so that

$$u_\varepsilon(x,t) \to u(x,t), \quad \text{in } L^2(\Omega \times [0,T]), \quad \text{as } \varepsilon \to 0^+,$$

with $u(x,t)$ a solution of (P_{HG}). Thereby, the convergence is strong in $L^2(\Omega \times [0,T])$. In the whole space $\mathbb{R}^N$ we use the appropriate $L^2_\rho(\mathbb{R}^N)$ and $H^{2m}_\rho(\mathbb{R}^N)$ weighted spaces. Note that the difficult issue, that we don not overcome at this stage, is whether the limit depends on the taken subsequence, in other words, if the limit as $\varepsilon \to 0^+$ provides a unique limit or many partial limits.

LEMMA 6.4.2. *Let $u_\varepsilon(x,t)$ be the unique global solution of the regularized problem* (6.1.1)*, then*

$$||u_\varepsilon(\cdot,t) - u(x,\cdot)||_{L^2(\Omega\times(0,t))} \to 0, \ \text{as } \varepsilon \to 0^+,$$

with $u(x,t)$ a solution of (P_{HG}), i.e.,

$$\iint_{\Omega\times(0,t)} \varphi_t u dxdt + (-1)^m \iint_{\Omega\times(0,t)} \nabla\varphi \left(f^n(u)\nabla\Delta^{m-1}u\right)\, dx\, dt = 0,$$

for all $\varphi \in C_0^\infty(\Omega \times (0,t)), t \in [0,T]$.

PROOF. Multiplying equation (6.1.1) by a test function $\varphi \in C_0^\infty(\Omega\times(0,t))$ and integrating by parts we get

$$\iint_{\Omega\times(0,t)} \varphi_t u_\varepsilon dxdt + (-1)^m \iint_{\Omega\times(0,t)} \nabla\varphi \left(\phi_\varepsilon(u)\nabla\Delta^{m-1}u_\varepsilon\right)\, dx\, dt = 0.$$

Substituting $\phi_\varepsilon(u) = f^n(\varepsilon) + (1-\varepsilon)f^n\left((\varepsilon^2 + u_\varepsilon^2)^{1/2}\right)$ into the latter equation we find,

$$\begin{aligned} &\iint_{\Omega\times(0,t)} \varphi_t u_\varepsilon dxdt + (-1)^m f^n(\varepsilon) \iint_{\Omega\times(0,t)} \nabla\varphi \cdot \nabla\Delta^{m-1}u_\varepsilon\, dx\, dt \\ &+ (-1)^m(1-\varepsilon) \iint_{\Omega\times(0,t)} f^n\left((\varepsilon^2 + u_\varepsilon^2)^{1/2}\right) \nabla\varphi \cdot \nabla\Delta^{m-1}u_\varepsilon\, dx\, dt = 0. \end{aligned} \tag{6.4.1}$$

Now, we focus on controlling the second term in (6.4.1). To do so, using the Hölder's inequality together with (6.3.3) in Proposition 6.3.1 we find ,

$$\begin{aligned}
&\left| f^n(\varepsilon) \iint_{\Omega\times(0,t)} \nabla\varphi\cdot\nabla\Delta^{m-1}u_\varepsilon\, dx\, dt \right| \\
&\leq f^n(\varepsilon)\left(\iint_{\Omega\times(0,t)} |\nabla\Delta^{m-1}u_\varepsilon|^2 dxdt\right)^{\frac12}\left(\iint_{\Omega\times(0,t)} |\nabla\varphi|^2\, dx\, dt\right)^{\frac12} \\
&\leq C f^{\frac n2}(\varepsilon)\left(f^n(\varepsilon)\iint_{\Omega\times(0,t)} |\nabla\Delta^{m-1}u_\varepsilon|^2\, dx\, dt\right)^{\frac12} \\
&\leq K f^{\frac n2}(\varepsilon)\to 0, \quad \text{as } \varepsilon\to 0^+
\end{aligned}$$

with K a positive constant. To control the third term we split the integration domain in the following sets

$$\mathcal{G}_{\varepsilon,\delta} := \{(x,t)\in\Omega\times(0,t) : |u_\varepsilon(x,t)|>\delta>0\},$$

and

$$\mathcal{B}_{\varepsilon,\delta} := \{(x,t)\in\Omega\times(0,t) : |u_\varepsilon(x,t)|\leq\delta\},$$

for any fixed arbitrarily small $\delta>0$. In the uniform non-degeneracy set $\mathcal{G}_{\varepsilon,\delta}$ it is clear that the limiting solution as $\varepsilon\to 0^+$ is a weak solution of (P_{HG}). Also, by parabolic regularity for the uniformly parabolic equation (6.1.1), we get that $u_{\varepsilon,t}$ and $\phi_\varepsilon(u_\varepsilon)\nabla\Delta^{m-1}u_\varepsilon$ converge in compact subsets of $\mathcal{G}=\mathcal{G}_{0,0}$. Thus, as it happens in [**21**] and [**8**] we obtain that the limit function $u(x,t)=\lim_{\varepsilon\to 0^+} u_\varepsilon(x,t)$ satisfies

$$\iint_{\mathcal{G}} \varphi_t u\; dxdt + (-1)^m \iint_{\mathcal{G}} \nabla\varphi\left(f^n(|u|)\nabla\Delta^{m-1}u\right) dxdt = 0. \tag{6.4.2}$$

Then, the limit function $u(x,t)$ is a solution to the Cauchy Problem (P_{HG}). Nevertheless, in the set of parabolic degeneracy $\mathcal{B}_{\varepsilon,\delta}$, we have to take $\varepsilon>0$ small enough and depending on δ. Indeed, let $0<\varepsilon\leq\delta$ fixed. Applying the Hölder's inequality to the third term in (6.4.1) in the set $\mathcal{B}_{\varepsilon,\delta}$ and using that f is a continuous monotone increasing function, we find

$$\begin{aligned}
&\left|\iint_{\mathcal{B}_{\varepsilon,\delta}} \nabla\varphi f^n\left((\varepsilon^2+u_\varepsilon^2)^{1/2}\right)\nabla\Delta^{m-1}u_\varepsilon dxdt\right| \\
&\leq C\left(\iint_{\mathcal{B}_{\varepsilon,\delta}} f^{2n}\left((\varepsilon^2+u_\varepsilon^2)^{1/2}\right)|\nabla\Delta^{m-1}u_\varepsilon|^2 dxdt\right)^{\frac12} \\
&\leq C f^{\frac n2}\left((\varepsilon^2+\delta^2)^{1/2}\right)\left(\iint_{\mathcal{B}_{\varepsilon,\delta}} f^{n}\left((\varepsilon^2+u_\varepsilon^2)^{1/2}\right)|\nabla\Delta^{m-1}u_\varepsilon|^2 dxdt\right)^{\frac12} \\
&\leq K f^{\frac n2}\left((\varepsilon^2+\delta^2)^{1/2}\right)\to 0,
\end{aligned}\tag{6.4.3}$$

provided $\delta\to 0$ as $\varepsilon\to 0^+$. Therefore, the integration over the set of degeneracy has no distinguishable effects respect to the integration over the sets $\mathcal{G}_{\varepsilon,\delta}$ in the final limit. Thus, the limit as $\varepsilon\to 0$ provides weak solutions to (P_{HG}). □

REMARK 6.4.1. *Due to the boundary conditions* (6.1.5) *the weak formulation* (6.4.2) *also holds for solutions to the FBP, so that our analysis is unable to distinguish solutions to the Cauchy problem from those to the FBP.*

Now we perform the limit when $n \to 0^+$. Let us notice that the estimate provided by (6.4.3) reflects the rate of convergence if we perform a second homotopic limit as $n \to 0$, together with $\varepsilon \to 0$, in the analytic regularization (6.1.3), in order to obtain weak solutions emanating from the polyharmonic heat equation (6.2.1).
To get such a functional convergence we need $n = n(\varepsilon) \to 0^+$ such that, for $\delta \approx \varepsilon$,

$$f^{n(\varepsilon)}(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0^+,$$

that is,

$$n(\varepsilon) \ln f(\varepsilon) \to -\infty, \quad \text{as } \varepsilon \to 0. \tag{6.4.4}$$

Hence, we need $n = n(\varepsilon)$ such that

$$n(\varepsilon) >> \frac{1}{|\ln f(\varepsilon)|}, \tag{6.4.5}$$

that will provide us with the convergence, at least in a weak sense, of solutions. Thus, under this hypotheses we arrive at a solution of the polyharmonic heat equation (6.2.1) as $\varepsilon, n(\varepsilon) \to 0$, written in the very-weak form,

$$\iint_{\Omega\times(0,t)} \varphi_t u \, dxdt + (-1)^m \iint_{\Omega\times(0,t)} \nabla\varphi \cdot \nabla\Delta^{m-1} u \, dxdt = 0.$$

Let us remark that this is not a full definition of weak solution since it just assumes a single integration by parts, so that performing the limit as $\varepsilon, n(\varepsilon) \to 0$ allows us to obtain, among other things a solution of (6.2.1) under the boundary conditions (6.1.5) (with n=0). It is clear now that applying this limiting argument in the integral identities does not allow us to ascertain any difference between CP-solutions and FBP-solutions.

Consequently, a stronger version of our homotopic arguments is indispensable to identify correctly the proper solutions to the Cauchy problem (P_{HG}).

Nonetheless, this homotopic approach provides us with estimates and bounds such as (6.4.4) and (6.4.5) which are necessary for a correct limiting process. Moreover, keeping in mind the oscillatory nature of the kernel $F(|y|)$ of the polyharmonic heat equation, inevitably, the proper solutions to (P_{HG}) are going to be oscillatory near the interface provided $n > 0$ is small enough.

6.4.1. Branching of solutions from the Polyharmonic heat equation.
Next we analyze the double limit as $n \to 0^+$ and $\varepsilon \to 0^+$. As a consequence we obtain the well-posedness of the equation (P_{HG}) through a homotopy deformation from solutions to the polyharmonic heat equation (6.2.1) (which are oscillatory) to solutions to problem (P_{HG}). To do so, we now consider the regularization

$$\psi_\varepsilon(u) = f^n\left((\varepsilon^2 + u)^{1/2}\right),$$

and therefore we will handle the following regularized equation

$$u_t = (-1)^{m-1} \nabla \cdot \left(\psi_\varepsilon(u) \nabla\Delta^{m-1} u\right), \tag{6.4.6}$$

with smooth compactly supported initial data. Due to parabolic estimates we may assume that $u_\varepsilon(x,t)$ decays exponentially at infinity. Moreover, now we take $n \to 0^+$, as the principal deformation parameter and then we will choose the appropriate

$$\varepsilon = \varepsilon(n) \to 0^+.$$

Next, we rewrite equation (6.4.6) as

$$u_t = -(-\Delta)^m u + (-1)^{m-1}\nabla \cdot \left([1-\psi_\varepsilon(u)]\nabla\Delta^{m-1}u\right),$$

that in terms of the fundamental solution for (6.2.1) can be written as

$$u(x,t) = \mathcal{H}(x,t) * u_0(x) + \int_0^t \nabla\mathcal{H}(x,t-s) * \Theta_{n,\varepsilon}(u(x,s))\nabla\Delta^{m-1}u(x,s)ds, \tag{6.4.7}$$

where $\Theta_{n,\varepsilon}(u) = 1-\psi_\varepsilon(u)$. The convergence to the well posed polyharmonic heat equation (6.2.1) will strongly depend on the weak limit of the second term of (6.4.7), i.e., on the behaviour of

$$\Theta_{n,\varepsilon}(u) = 1-\psi_\varepsilon(u) = 1 - f^n\left((\varepsilon^2+u^2)^{1/2}\right) \to 0, \quad \text{as } n, \varepsilon(n) \to 0^+. \tag{6.4.8}$$

Thus, to carry out such a branching analysis we need to verify the following expansion:

$$\Theta_{n,\varepsilon}(u) = -n \ln f\left((\varepsilon^2+u^2)^{1/2}\right)(1+o(n)), \quad \text{as } n, \to 0^+. \tag{6.4.9}$$

on a fixed family of uniformly bounded smooth solutions $\{u_\varepsilon(x,t)\}$. Note that, checking (6.4.9) in the sets $\mathcal{B}_{\varepsilon,\delta}$, i.e., where $u \approx 0$, requires the condition

$$n\left|\ln f\left(\varepsilon(n)\right)\right| \to 0, \quad \text{as } n \to 0^+. \tag{6.4.10}$$

This will be the principal assumption on the parameter $\varepsilon(n)$ and its relation with n, in order to guarantee such convergence of solutions.

Proof of Theorem 6.1.1. Under the condition (6.4.10) we perform a branching analysis following the steps performed in [**8**]. Substituting (6.4.9) in (6.4.7), we find,

$$\begin{aligned} u(x,t) =& \mathcal{H}(x,t) * u_0(x) \\ &-n\int_0^t \nabla\mathcal{H}(x,t-s) * \ln f\left((\varepsilon^2+u^2)^{1/2}\right)\nabla\Delta^{m-1}u(x,s)ds + o(n^2). \end{aligned} \tag{6.4.11}$$

Now, we take

$$u = u_{PH}(x,t) + n\varphi + o(n),$$

with $u_{PH}(x,t)$ a solution to the polyharmonic heat equation (6.2.1) and φ to be determined. Thus, substituting into (6.4.11), and omitting terms of high order we obtain

$$\begin{aligned} &u_{PH}(x,t) + n\varphi = \mathcal{H}(x,t) * u_0(x) \\ &-n\int_0^t \nabla\mathcal{H}(x,t-s) * \ln f\left((\varepsilon^2 + u_{PH}^2(x,s) + 2nu_{PH}\varphi + n^2\varphi)^{1/2}\right)\nabla\Delta^{m-1}(u_{PH}(x,s))ds. \end{aligned}$$

Passing to the limit as $n \to 0^+$ we get the following expression for the error function

$$\varphi = \int_0^t \nabla\mathcal{H}(x,t-s) * \ln f\left(|u_{PH}|\right)\nabla\Delta^{m-1}u_{PH}ds. \tag{6.4.12}$$

The asymptotic expansion assumes that (6.4.12) is always finite, i.e.

$$\ln f\left(|u_{PH}|\right) \in L^1_{loc}(\mathbb{R}^N),$$

for any $t>0$, so $f(|u_{PH}|)$ does not have zeros with an exponential decay in some neighbourhood. In particular, this is true if the solutions have transversal zeros. Observe that to obtain (6.4.9) from (6.4.8), we have to use the expansion for small $n>0$,

$$1-|f|^n \equiv 1-e^{n\ln|f|} = 1-(1+n\ln|f|+\ldots) = n\ln|f|+\ldots, \tag{6.4.13}$$

which is true pointwise on any set $\{f \geq c_0\}$ for an arbitrarily small fixed constant $c_0>0$. However, in a small neighborhood of any zero of $f(|u_{PH}|)$, the expansion (6.4.13) is no longer true. Nevertheless, it remains true in a weak sense provided that this zero is sufficiently transversal in a natural sense, i.e.,

$$\frac{1-|f|^n}{n} \rightharpoonup -\ln|f|, \quad \text{as } n\to 0^+$$

in $L^\infty_{\rm loc}$. Although this fact is rather plausible, as it is noted in [**8**], there is not a rigorous proof for general solutions to the polyharmonic heat equation. Therefore, we include such assumptions in our argument.

Finally, we have to check that the perturbation $\Theta_{n,\varepsilon}(u)$ is small, which is guaranteed by the following.

(1) At one hand, thanks to the uniform estimate (6.3.4), using the Young inequality for convolutions, we find that $\Theta_{n,\varepsilon}(u)\to 0$ as $n,\varepsilon(n)\to 0^+$ for the domain $\{|u|\geq t_1\}$ with

$$|\ln t| \leq c f^{\frac{n}{2}}(t), \quad \text{with } t\geq t_1,$$

for some constant $c>0$.

(1b) Observe that, in a similar way as above, thanks to the uniform estimate for h_ε in Proposition 6.3.1, we find that $\Theta_{n,\varepsilon}(u)\to 0$ as $n,\varepsilon(n)\to 0^+$ for the domain $\{|u|\geq t_2\}$ with

$$|\ln t| \leq c f^{n}(t), \quad \text{with } t\geq t_2.$$

(2) On the other hand, consider the integral equality (6.4.7) in the domain where

$$\mathcal{D}_{i,\varepsilon} \equiv \{\varepsilon^2 \leq \varepsilon^2+u^2 \leq t_i\}, \quad i=1,2.$$

The maximal singularity of the term $\ln f\left((\varepsilon^2+u^2)^{1/2}\right)$ in the domain $\mathcal{D}_{i,\varepsilon}$ is achieved when $u=0$. Therefore, it is of order $O(\ln f(\varepsilon))$ and, hence, the perturbation term has order at most $O(n\ln f(\varepsilon))$. Then, because of (6.4.10) we conclude

$$O(n\ln f(\varepsilon))\to 0, \quad \text{as } n\to 0^+.$$

□

Let us stress that the representation $u=u_{PH}(x,t)+n\varphi+o(n)$ provided by Theorem 6.1.1, requires the convergence of (6.4.12) as $n\to 0^+$ which is difficult to verify for arbitrary solutions to the polyharmonic heat equation (6.2.1). Nevertheless, thanks to the regularity of solutions such an integral divergence due to the formation of flat zeros can occur at a finite number of points, so it is expected at least almost everywhere. As it happens in the case $m=2$ and $f(t)=t$, see [**8**], solutions to (P_{HG}) are those which can be deformed as $n\to 0^+$ through the analytic path $\psi_\varepsilon(u)$ to the unique solution to the polyharmonic heat equation with same initial data. Therefore, according to our development, a suitable setting of the Cauchy problem for the high order problem (P_{HG}) requires the whole set of solutions $\{u(x,t):\ n>0\}$ or the two-parameter set $\{u_\varepsilon(x,t):\ n>0, \varepsilon>0\}$ of regularized solutions. Hence, this approach results useless to treat an individual problem of type (P_{HG}) for a fixed $n > 0$. Nonetheless, it provides qualitative properties for solutions to problem (P_{HG}) inherited from those solutions to the polyharmonic heat equation (6.2.1). Finally, we observe that, due to the nature of the nonlinear term $f(\cdot)$ we are unable to provide a conclusive answer to whether

$$\limsup u_\varepsilon(x,t) = \liminf u_\varepsilon(x,t), \quad \text{as } \varepsilon \to 0^+.$$

In the case $m = 2$ and $f(t) = t$, studied in [**8**], the proof of such equality relies on the homogeneity properties of the non linear term $f(t) = t$. In fact, the proof follows studying an auxiliary problem independent of ε obtained by means of a scaling in the space variables for the regularized problem (6.1.1). Therefore these arguments automatically extends to the case of consider $f(t) = t^\kappa$ for $\kappa > 0$. Hence, as the one-variable homogeneous functions are such a power functions, this ideas does not work when one considers a general nonlinearity.

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